2.997 Decision-Making in Large-Scale Systems
MIT, Spring 2004 Handout #24

Problem Set 5

1. Consider the average-cost LP:

$$\begin{array}{ll} \max_{\lambda,h} & \lambda \\ \text{s.t.} & \lambda e + h \leq Th, \end{array}$$

where $Th = \min_u g_u + P_u h$.

   (a) Suppose that there is a unique optimal policy u^*, with a single class of recurrent states $\mathcal{R}$. Show that the optimal solution of the LP is given by $(\lambda^*, \bar{h})$, where λ^* is the optimal average cost and $\bar{h}(x) = h^*(x)$ for all $x \in \mathcal{R}$.

   (b) Provide an example of an MDP such that there is an optimal solution $\bar{h}$ to the LP such that at least one greedy policy with respect to $\bar{h}$ is not optimal.

   (c) Propose an algorithm based on linear programming for computing the differential cost function h^*.

2. Let u_h denote the average-cost greedy policy with respect to h, i.e., $u_h = \underset{u}{\operatorname{argmin}}(g_u + P_u h)$. Let λ_h denote its average cost, and π_h denote its stationary state distribution. Show that

$$\lambda_h - \lambda^* = \pi_h^T (Th - h - \lambda^*) \leq \|Th - h - \lambda^*\|_{1,\pi_h}.$$

3. Let h be such that

$$\alpha Th \geq h + \lambda^* e,$$

for some $\alpha < 1$. Let $h_\alpha = \min_u (I - \alpha P_u)^{-1}(\alpha g_u - \lambda^*)$.

   (a) Show that

$$c^T(Th - h - \lambda^* e) \leq c^T(h_\alpha - \Phi r) + \frac{1-\alpha}{\alpha} c^T h_\alpha, 0.$$

   (b) Show that $\lim_{\alpha \uparrow 1}(1-\alpha)c^T \max(h_\alpha, 0) = 0$.

   (c) Suppose that there is v such that $\alpha \max_u P_u \Phi v \leq \beta \Phi v$, for some $\beta < 1$ and all v. Denote by $\tilde{r}$ the optimal solution of the LP

$$\begin{array}{ll} \max_r & c^T \Phi r \\ \text{s.t.} & \alpha T \Phi r \geq \Phi r + \lambda^* e. \end{array}$$

   Show that

$$c^T(T\Phi\tilde{r} - \Phi\tilde{r} - \lambda^* e) \leq \frac{2c^T \Phi v}{1-\beta} \min_r \|h_\alpha - \Phi r\|_{\infty, 1/\Phi v} + \frac{1-\alpha}{\alpha} c^T h_\alpha.$$

   (d) Suppose that $\Phi v = e$ for some v. Let $R(\lambda)$ denote the set of optimal solutions to

$$\begin{array}{ll} \max_r & c^T \Phi r \\ \text{s.t.} & \alpha T \Phi r \geq \Phi r + \lambda e. \end{array}$$

   Let λ and $\bar{\lambda}$ be arbitrary. Show that if u_r is a greedy policy with respect to Φr, for some $r \in R(\lambda)$, then it is also a greedy policy with respect to $\Phi \bar{r}$, for some $\bar{r} \in R(\bar{\lambda})$.