Introduction to Numerical Analysis for Engineers

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Numerical Differentiation

Taylor Series

$$f(x) = f_0 + \frac{\Delta f_0}{h}(x - x_0) + \frac{\Delta^2 f_0}{2!h^2}(x - x_0)(x - x_1) \cdots$$

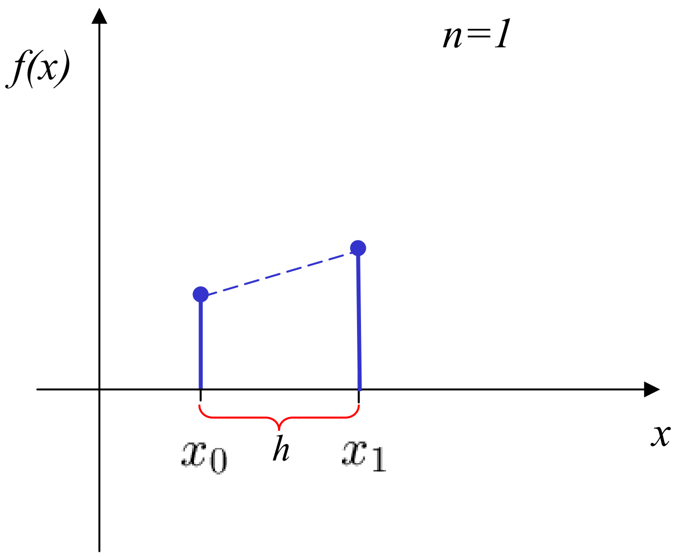
$$= + \frac{\Delta^n f_0}{2!h^2}(x - x_0)(x - x_1) \cdots (x - x_{n-1} + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0) \cdots (x - x_n)$$

First order

$$n = 1$$

$$f(x) = f_0 + \frac{\Delta f_0}{h}(x - x_0) + \frac{f''(\xi)}{2!}(x - x_0)(x - x_1)$$

$$f'(x) = \frac{\Delta f_0}{h} + O(h) = \frac{1}{h}(f_1 - f_0) + O(h)$$



Numerical Differentiation

Second order

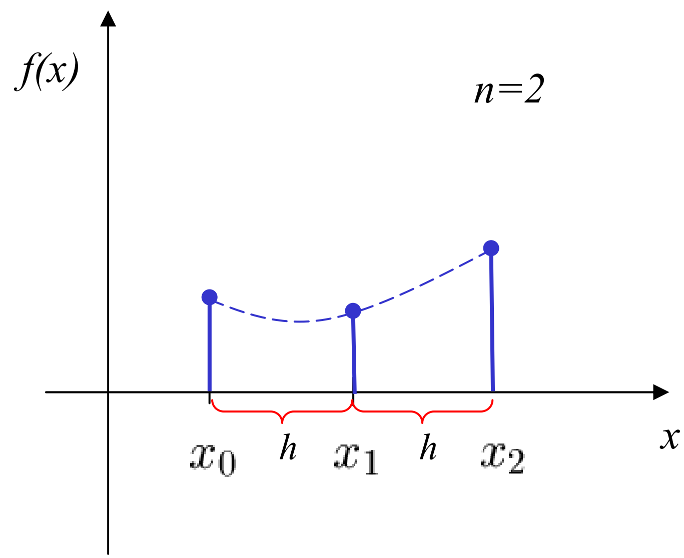
$$n = 2$$

$$f(x) = f_0 + \frac{\Delta f_0}{h}(x-x_0) + \frac{\Delta^2 f_0}{2!h^2}(x-x_0)(x-x_1) + \frac{f'''(\xi)}{3!}(x-x_0)(x-x_1)(x-x_2)$$

$$f'(x) = \frac{\Delta f_0}{h} + \frac{\Delta^2 f_0}{h}(x - x_0) + \frac{\Delta^2 f_0}{h}(x - $$

$$\begin{aligned} f'(x_0) &= \frac{f_1 - f_0}{h} - \frac{1}{2h}(f_2 - 2f_1 + f_0) + O(h^2) \\ &= \frac{2f_1 - 2f_0 - f_2 + 2f_1 - f_0}{2h} + O(h^2) \\ &= \boxed{\frac{1}{h}(-\frac{3}{2}f_0 + 2f_1 - \frac{1}{2}f_2) + O(h^2)} \end{aligned}$$

$$\begin{aligned} f'(x_1) &= \frac{f_1 - f_0}{h} + \frac{1}{2h}(f_2 - 2f_1 + f_0) + O(h^2) \\ &= \boxed{\frac{1}{2h}(f_2 - f_0) + O(h^2)} \end{aligned}$$



Second Derivatives

n=2 $\quad f''(x_0) = \frac{\Delta^2 f_0}{h^2} + O(h) = \boxed{\frac{1}{h^2}(f_0 - 2f_1 + f_2) + O(h)}$ Forward Difference

n=3 $\quad f''(x_1) = \boxed{\frac{1}{h^2}(f_0 - 2f_1 + f_2) + O(h^2)}$ Central Difference

Numerical Integration

Lagrange Interpolation

$$I = \int_a^b f(x)dx$$

$$f(x) \simeq p(x) = \sum_{k=0}^{n} L_k(x) f(x_k)$$

$$L_k(x) = \frac{(x-x_0)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

Equidistant Sampling

$$\begin{aligned} x_k &= x_0 + kh \\ x &= x_0 + sh \end{aligned}$$

$$L_k(x) = \frac{s(s-1)(s-2)\cdots(s-k+1)(s-k-1)\cdots(s-n)}{k(k-1)(k-2)\cdots(1)(-1)\cdots(k-n)}$$

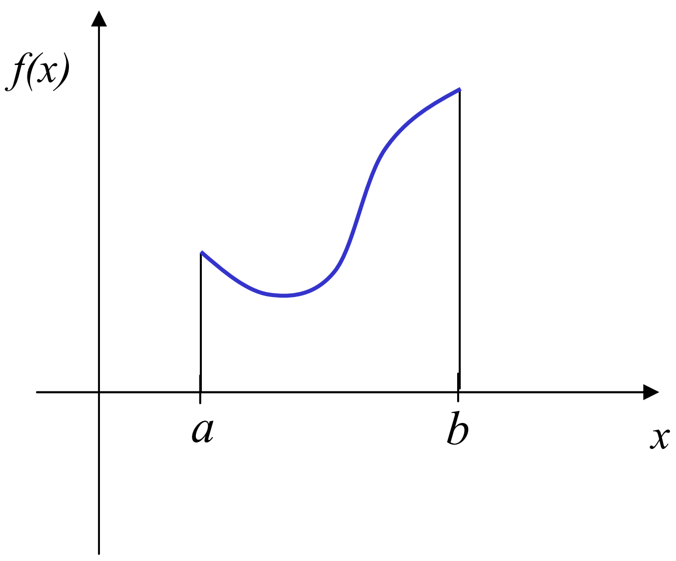
$$I = \int_a^b f(x)dx \simeq \int_{x_0}^{x_n} p(x)dx = h\sum_{k=0}^{n} f(x_k)\int_0^n L_k(s)ds = nh\sum_{k=0}^{n} f(x_k) C_k^n$$

Integration Weights (Cote's Numbers)

$$C_k^n = \frac{1}{n}\int_0^n L_k(s)ds$$

Properties

$$\begin{aligned} C_k^n &= C_{n-k}^n \\ \sum_{k=0}^{n} C_k^n &= 1 \end{aligned}$$



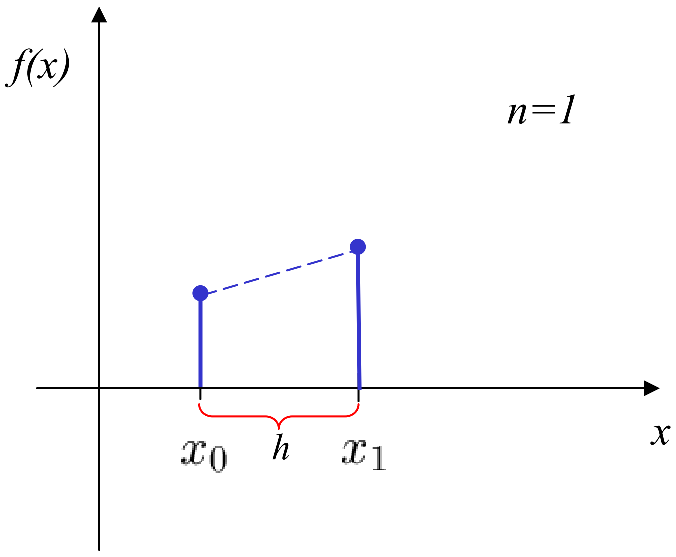
Numerical Integration

$n = 1$ **Trapezoidal Rule**

$$k=0: \quad C_0^1 = \int_0^1 \frac{s-1}{-1} ds = 1 - 1/2 = 0.5$$

$$k=1: \quad C_1^1 = \int_0^1 \frac{s}{1} ds = 1/2 = 0.5$$

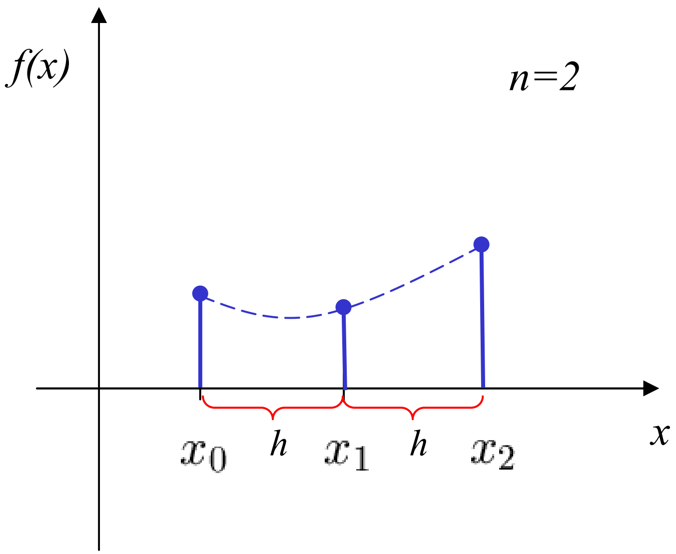
$$\int_{x_0}^{x_1} f(x)dx \simeq 1 \cdot (x_1 - x_0)\left(\frac{1}{2}f(x_0) + \frac{1}{2}f(x_1)\right) = \frac{1}{2}(x_1 - x_0)(f(x_0) + f(x_1))$$



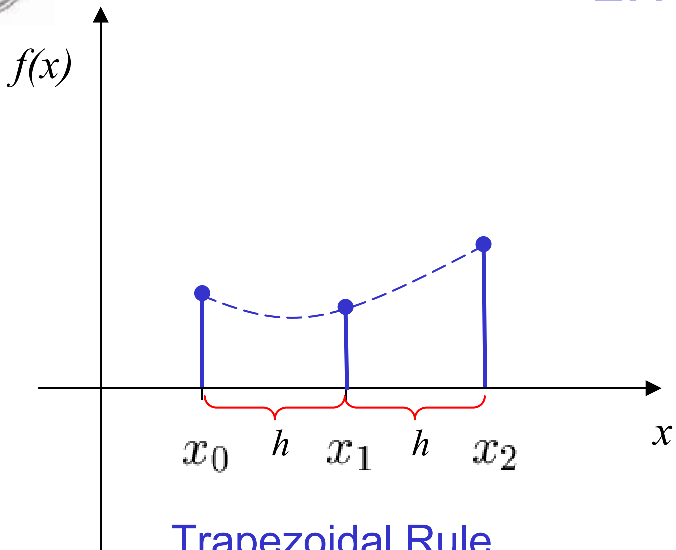
$n = 2$ **Simpson's Rule**

$$\begin{aligned}
k=0: \quad C_0^2 &= \frac{1}{2}\int_0^2 \frac{(s-1)(s-2)}{(-1)(-2)} ds \\
&= \frac{1}{4}\int_0^2 (s^2 - 3s + 2) ds \\
&= \frac{1}{4}\left[\frac{s^3}{3} - \frac{3s^2}{2} + 2s\right] \\
&= \frac{1}{4}\left[\frac{8}{3} - \frac{12}{2} + 4\right] = \frac{1}{4}\cdot\frac{4}{6} = \frac{1}{6} \\
k=1: \quad C_1^2 &= \frac{1}{2}\int_0^2 \frac{s(s-2)}{(1)(-1)} ds \\
&= \frac{1}{2}\int_0^2 (2s - s^2) ds \\
&= \frac{1}{2}\left[s^2 - \frac{s^3}{3}\right] \\
&= \frac{1}{2}\left[4 - \frac{8}{3}\right] = \frac{2}{3} \\
k=2: \quad C_2^2 &= C_0^2 = \frac{1}{6}
\end{aligned}$$

$$\int_{x_0}^{x_1} f(x)dx \simeq 2h\frac{1}{6}(f(x_0) + 4f(x_1) + f(x_2)) = \frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2))$$



Numerical Integration
Error Analysis



Trapezoidal Rule

$$n = 1$$

$$e(x) = p(x) - f(x) = -\frac{f''(\xi)}{2}(x - x_0)(x - x_1)$$

Local Absolute Error

$$|\epsilon| = \left| -\int_{x_0}^{x_1} \frac{f''(\xi)}{2}(x - x_0)(x - x_1)dx \right|$$

$$\leq -\frac{\max|f''|}{2}(x - x_0)(x - x_1)dx$$

$$= \frac{\max|f''|}{2} h^3 \int_0^1 s(s-1)ds = \frac{h^3}{12}\max|f''| \simeq O(h^3)$$

N Intervals

$$E = \sum_{i=1}^{N} \epsilon_i \leq \frac{h^3}{12}\sum_{i=1}^{N} \max|f''| \leq \frac{Nh^3}{12}\max|f''| = \frac{(b-a)h^2}{12}\max|f''| \simeq O(h^2)$$

Simpson's Rule

$$n = 2$$

$$I = \int_{x_{m-1}}^{x_{m+1}} f(x)dx \simeq \frac{h^3}{3}\left[f_{m-1} + 4f_m + f_{m+1}\right]$$

Local Error

$$\epsilon_m = -\int_{x_{m-1}}^{x_{m+1}} \frac{f'''(\xi)}{6}(x - x_{m-1})(x - x_m)(x - x_{m+1})dx \simeq O(h^4)$$

Global Error

$$E = O(h^3)$$

$$x_m = 0,\ x_{m-1} = -h,\ x_{m+1} = h$$

$$f(x) = f_0 + x f_0' + \frac{x^2}{2} f_0'' + \frac{x^3}{3!} f_0''' + O(h^4)$$

$$\begin{aligned} I &= \int_{-h}^{h} f(x)dx \\ &= f_0 \int_{-h}^{h} x dx + \frac{f_0''}{2}\int_{-h}^{h} x^2 dx + \frac{f_0'''}{6}\int_{-h}^{h} x^3 dx + O(h^4) \\ &= 2hf_0 + 0 + \frac{h^3}{3} f_0'' + 0 + O(h^5) \end{aligned}$$

$$f_0'' = \frac{1}{h^2}(f_{-1} - 2f_0 + f_1) + O(h^2)$$

$$\begin{aligned} I &= 2hf_0 + \frac{h}{3}(f_{-1} - 2f_0 + f_1) + O(h^5) \\ &= \frac{h}{3}(f_{-1} + 4f_0 + f_1) + O(h^5) \end{aligned}$$

Local Error: $O(h^5)$

Global Error: $E = O(h^4)$