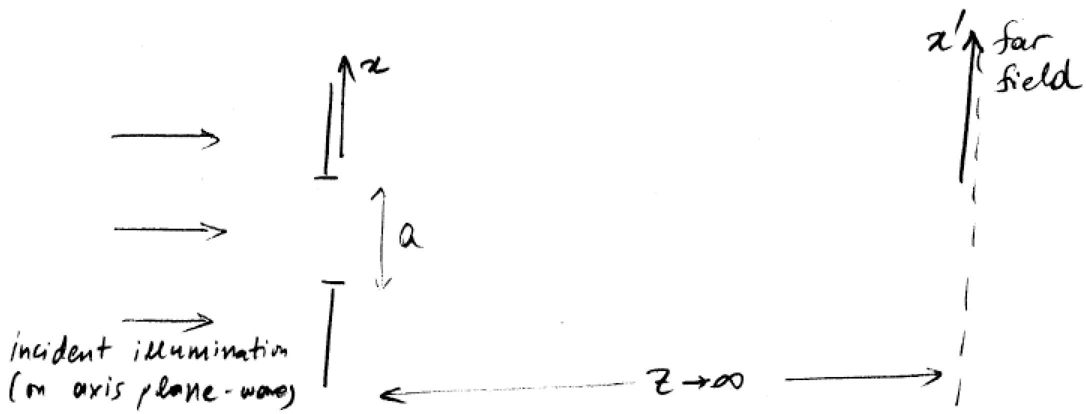
1. What is the Fraunhofer diffraction pattern of a 1-D slit of size a?

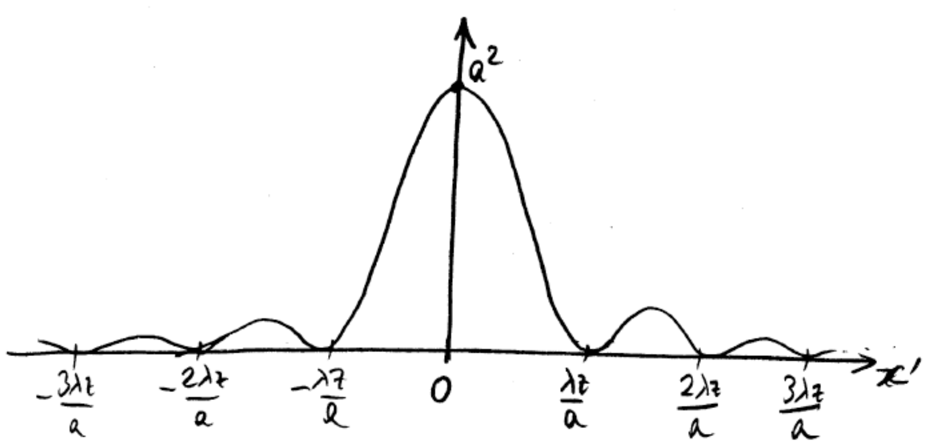


Slit description (1D): $f(x) = \text{rect}\left(\frac{x}{a}\right)$

Fourier transform of slit: $\mathcal{F}(u) = a\text{sinc}(au)$

Diffracted far field: $g(x') = e^{i\pi\frac{x'^2+y'^2}{\lambda z}} \times \mathcal{F}\left(\frac{x'}{\lambda z}\right)$

Fraunhofer diffraction pattern (intensity): $|g(x')|^2 = a^2\text{sinc}^2\left(\frac{ax'}{\lambda z}\right)$

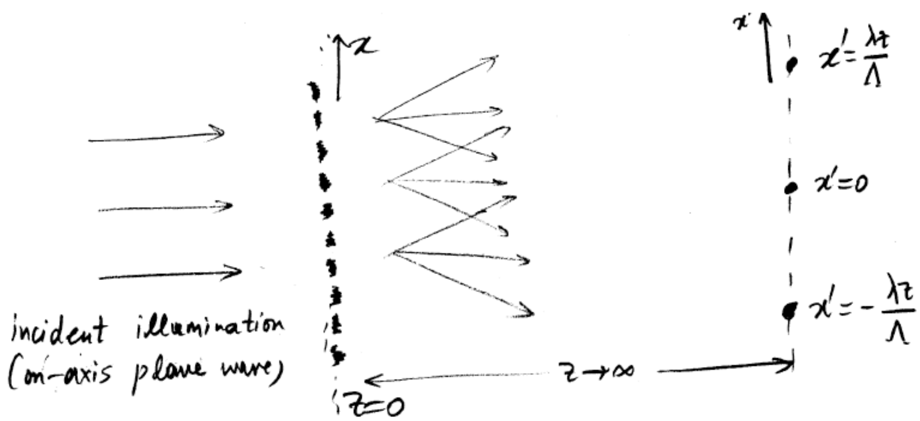


2. What is the Fraunhofer diffraction pattern of this sinusoidal amplitude grating, where Λ is the grating period?

Solution:

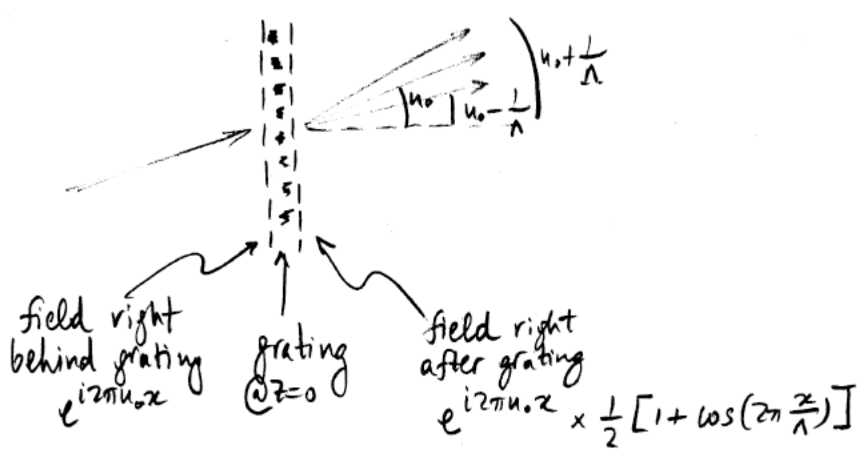
$$\begin{aligned}
f(x) &= \frac{1}{2}\left[1+\cos\left(2\pi\frac{x}{\Lambda}\right)\right] \\
&= \text{DC term (0th order)} + \text{diffracted orders} \\
&= \underbrace{\frac{1}{2}}_{\substack{\text{plane wave,}\\ u=0}} + \underbrace{\frac{1}{4}e^{i2\pi\frac{x}{\Lambda}}}_{\substack{\text{plane wave,}\\ u=\frac{1}{\Lambda}}} + \underbrace{\frac{1}{4}e^{-i2\pi\frac{x}{\Lambda}}}_{\substack{\text{plane wave,}\\ u=-\frac{1}{\Lambda}}} \\
\mathcal{F}(u) &= \frac{1}{2}\delta(u) + \frac{1}{4}\delta\left(u-\frac{1}{\Lambda}\right) + \frac{1}{4}\delta\left(u+\frac{1}{\Lambda}\right) \\
g(x') &= e^{i\pi\frac{x'^2+y'^2}{\lambda z}}\left[\frac{1}{2}\delta\left(\frac{x'}{\lambda z}\right) + \frac{1}{4}\delta\left(\frac{x'}{\lambda z}-\frac{1}{\Lambda}\right) + \frac{1}{4}\delta\left(\frac{x'}{\lambda z}+\frac{1}{\Lambda}\right)\right]
\end{aligned}$$

Note: without being too rigorous mathematically, we treat the intensity corresponding to the δ-function field as a "very bright and sharp" spot.

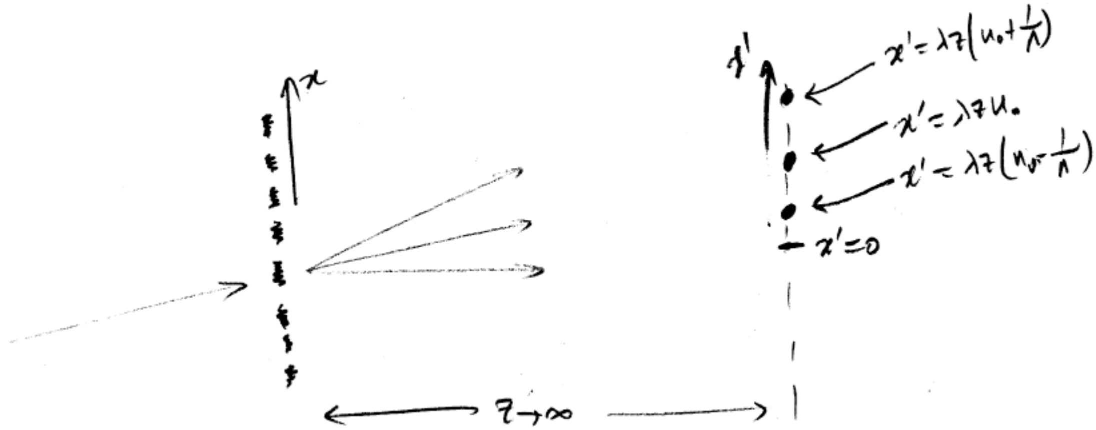


3. How does the result of problem 2 change if the illumination is a plane wave incident at angle θ_0 with respect to the optical axis? ($\theta_0 << 1$)

Solution: Let $u_0 = \dfrac{\sin\theta_0}{\lambda}$, so the plane wave is $e^{i2\pi u_0 x}$ (at $z=0$).



$$\mathcal{F}\left\{e^{i2\pi u_0 x} \times \frac{1}{2}\left[1+\cos\left(2\pi\frac{x}{\Lambda}\right)\right]\right\} = \frac{1}{2}\delta(u-u_0)+\frac{1}{4}\delta\left(u-u_0-\frac{1}{\Lambda}\right)+\frac{1}{4}\delta\left(u-u_0+\frac{1}{\Lambda}\right)$$



4. What is the Fraunhofer pattern of this truncated sinusoidal amplitude grating? Assume that $a >> \Lambda$.

$$f(x) = \frac{1}{2}\left[1+\cos\left(2\pi\frac{x}{\Lambda}\right)\right]\text{rect}\left(\frac{x}{a}\right)$$

Solution:

$$f_1(x) = \frac{1}{2}\left[1+\cos\left(2\pi\frac{x}{\Lambda}\right)\right] \quad \rightarrow \quad \mathcal{F}_1(u) = \frac{1}{2}\delta(u)+\frac{1}{4}\delta\left(u-\frac{1}{\Lambda}\right)+\frac{1}{4}\delta\left(u+\frac{1}{\Lambda}\right)$$

$$f_2(x) = \text{rect}\left(\frac{x}{a}\right) \quad \rightarrow \quad \mathcal{F}_2(u) = a\text{sinc}(au)$$

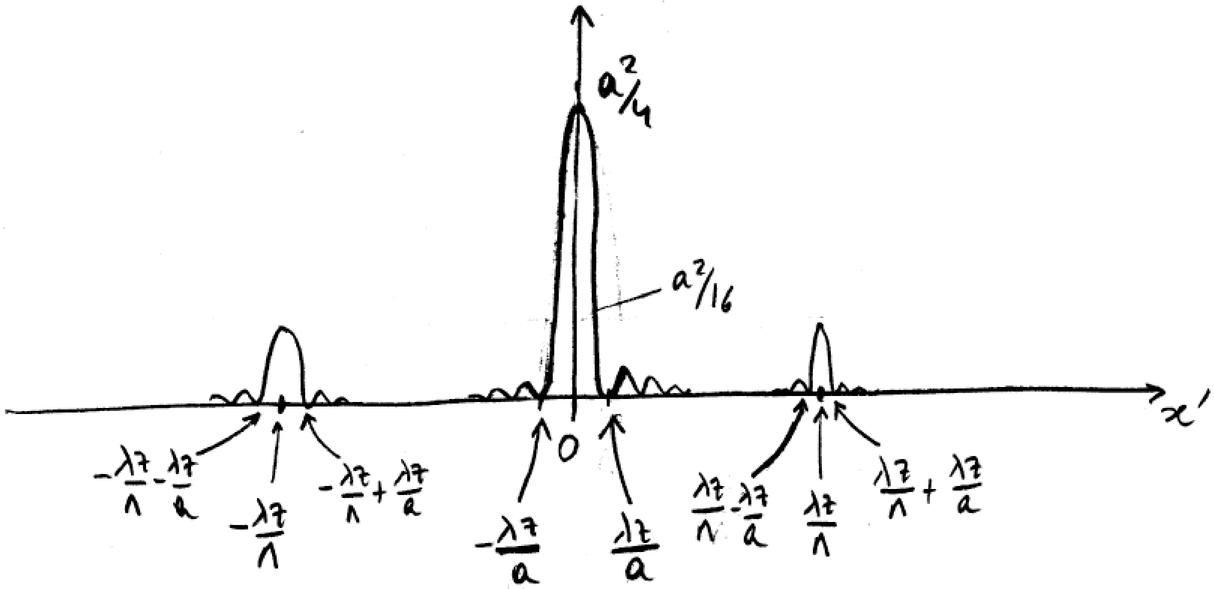
According to the convolution theorem, $\mathcal{F}\{f_1(x)\cdot f_2(x)\} = \mathcal{F}_1(u)\otimes\mathcal{F}_2(u)$.

Recall that: $\delta(u-u_0)\otimes A(u) = \int_{-\infty}^{\infty}\delta(u-u_0)A(u'-u)du = A(u'-u_0)$

$$\begin{aligned}\mathcal{F}_1(u)\otimes\mathcal{F}_2(u) &= \left[\frac{1}{2}\delta(u)+\frac{1}{4}\delta\left(u-\frac{1}{\Lambda}\right)+\frac{1}{4}\delta\left(u+\frac{1}{\Lambda}\right)\right]\otimes a\text{sinc}(au)\\ &= \frac{1}{2}a\text{sinc}(au)+\frac{1}{4}a\text{sinc}\left(a\left(u-\frac{1}{\Lambda}\right)\right)+\frac{1}{4}a\text{sinc}\left(a\left(u+\frac{1}{\Lambda}\right)\right)\end{aligned}$$

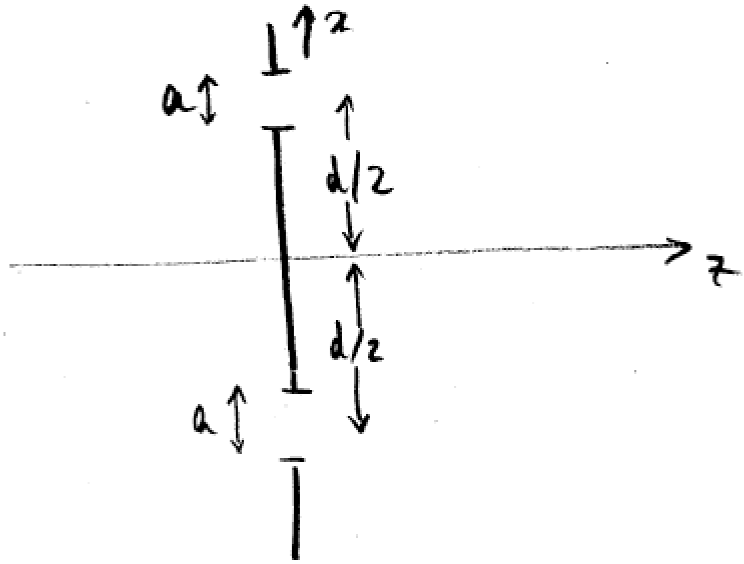
<u>Note</u>: When you take $||^2$, cross-terms can be ignored. Why?

$$\underbrace{|y(x')|^2}_{\substack{\text{Fraunhofer}\\\text{pattern}\\\text{(intensity)}}} \simeq \frac{a^2}{4}\text{sinc}^2\left(\frac{ax'}{\lambda z}\right)+\frac{a^2}{16}\text{sinc}^2\left(a\left(\frac{x'}{\lambda z}-\frac{1}{\Lambda}\right)\right)+\frac{a^2}{16}\text{sinc}^2\left(a\left(\frac{x'}{\lambda z}+\frac{1}{\Lambda}\right)\right)$$



Fraunhofer pattern of truncated grating

5. What is the Fraunhofer diffraction pattern of two identical slits (width a) separated by a distance $d >> a$?

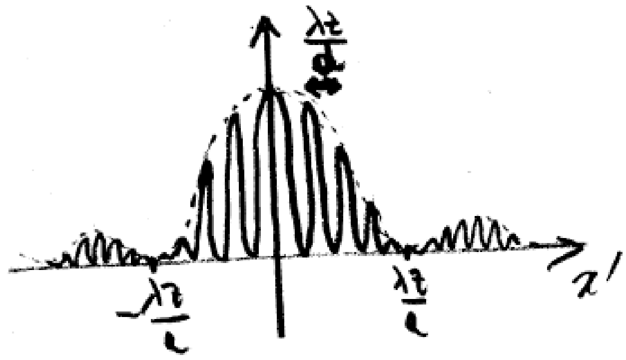


$$f(x) = \mathrm{rect}\left(\frac{x - \frac{d}{2}}{a}\right) + \mathrm{rect}\left(\frac{x + \frac{d}{2}}{a}\right)$$

Use the scaling and shift theorems, and linearity:

$$\begin{aligned}\mathcal{F}(u) &= a\mathrm{sinc}(au)e^{-i2\pi u\frac{d}{2}} + a\mathrm{sinc}(au)e^{i2\pi u\frac{d}{2}} \\ &= 2a\mathrm{sinc}(au)\cos(\pi u d)\end{aligned}$$

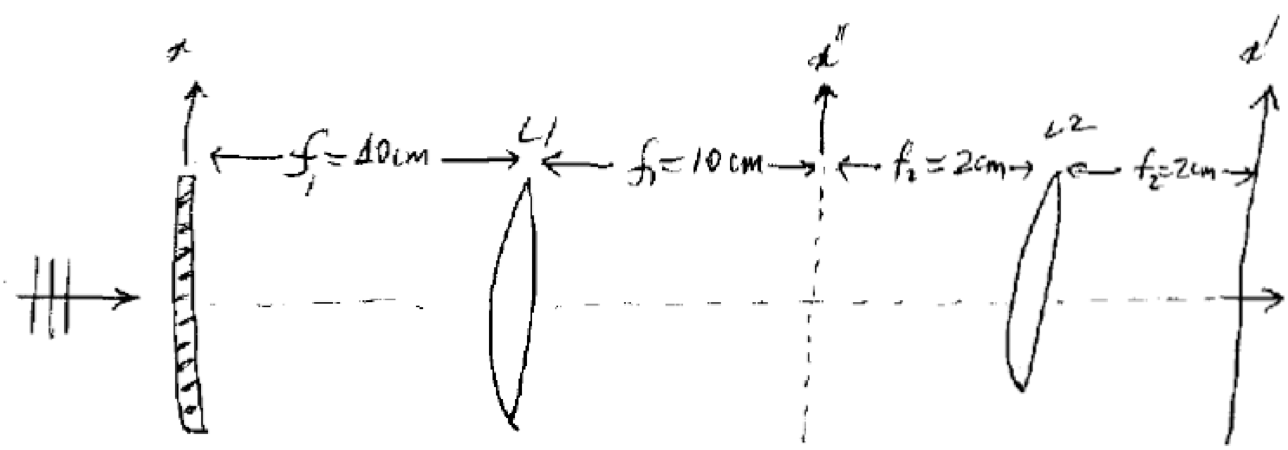
$$|g(x')|^2 = 4a^2 \text{sinc}^2\left(\frac{ax'}{\lambda z}\right)\cos\left(\frac{\pi x' d}{\lambda z}\right)$$



'modulated' sinc pattern

6. In the 4F system shown below, the sinusoidal transparency $t(x)$ is illuminated by a monochromatic plane wave on-axis, at wavelength $\lambda = 1\mu$m. Describe quantitatively the fields at the Fourier plane (x'') and the output plane (x').

$$t(x) = \frac{1}{2}\left[1 + \cos\left(2\pi \frac{x}{10\mu m}\right)\right]$$

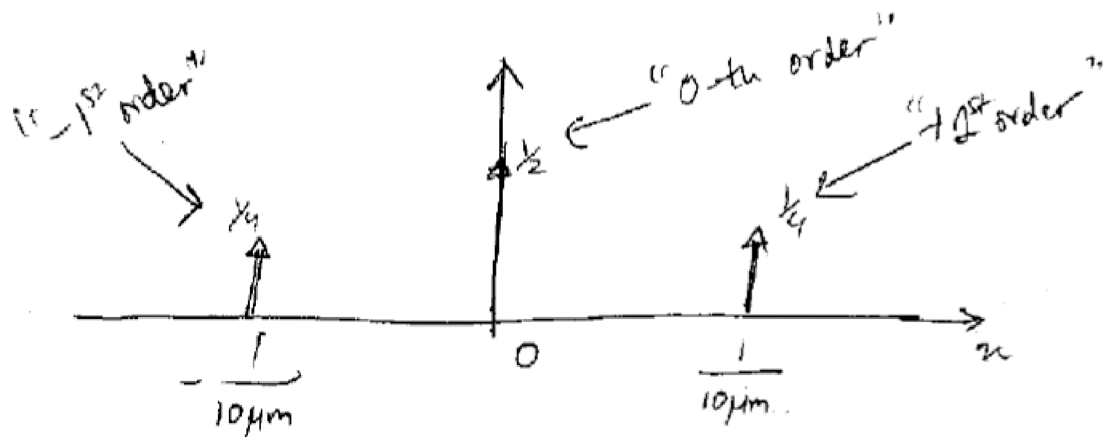


Solution: The field immediately past the transparency is produced by the on-axis plane wave multiplied by $t(x)$, the transmission function of the transparency.

$$g_{\text{in}}(x) = 1 \cdot t(x)$$

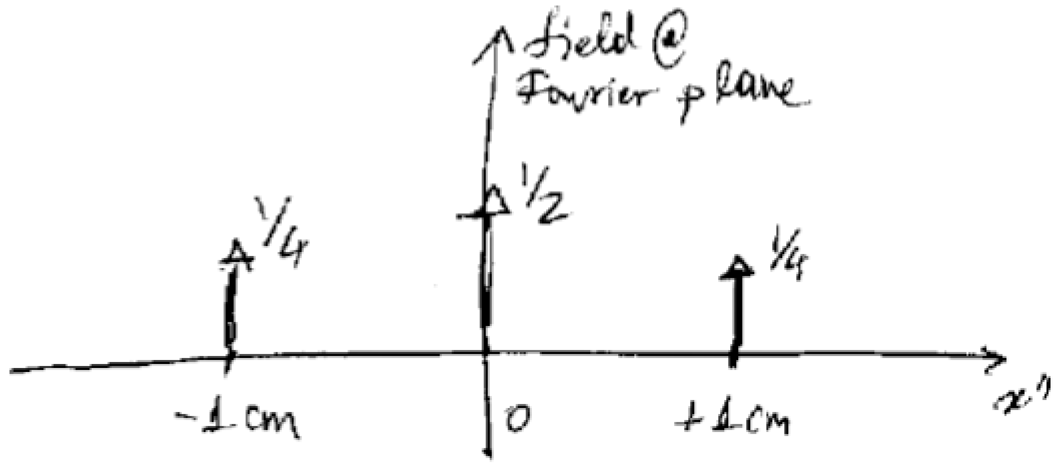
The field at the Fourier plane is $G_{in}\left(\frac{x''}{\lambda f_1}\right)$ where $G_{in}(u)$ is the Fourier transform of $g_{in}(x')$, i.e.

$$G_{in}(u) = \frac{1}{2}\left[\delta(u) + \frac{1}{2}\delta\left(u - \frac{1}{10\mu m}\right) + \frac{1}{2}\delta\left(u + \frac{1}{10\mu m}\right)\right]$$



The field at the Fourier plane will consist of three peaks corresponding to the three $\delta-$functions of the Fourier transform. The locations of these peaks are found as follows:

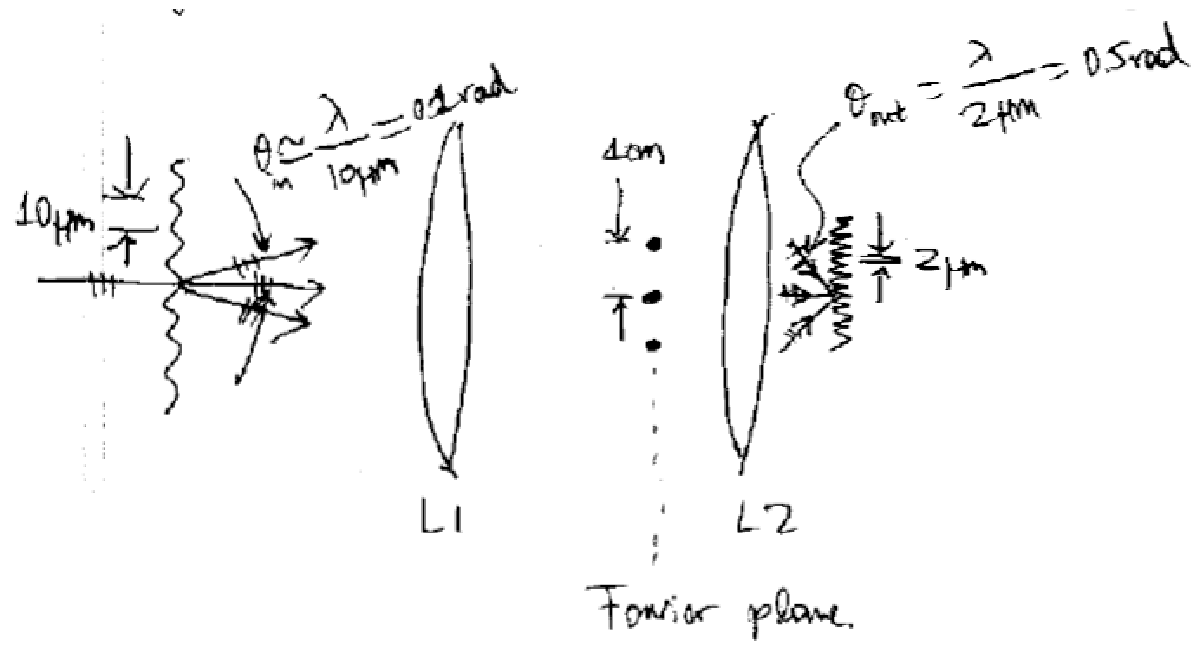
$$\begin{aligned}
&\text{+1st order:} \quad && u = \frac{1}{10\mu\text{m}} \Rightarrow \frac{x''}{\lambda f_1} = \frac{1}{10\mu\text{m}} \Rightarrow x'' = \frac{\lambda f_1}{10\mu\text{m}} = \frac{1\mu\text{m} \times 10\text{cm}}{10\mu\text{m}} = 1\text{cm} \\
&\text{0th order:} \quad && x'' = 0 \\
&\text{-1st order:} \quad && x'' = \cdots = -1\text{cm}
\end{aligned}$$



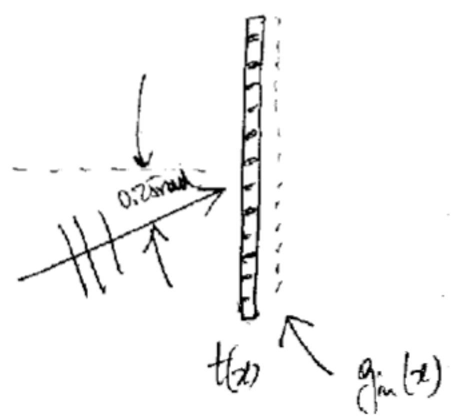
The field at the output plane is:

$$g_{\text{in}}\left(-\frac{f_1}{f_2}x'\right) = g_{\text{in}}\left(-\frac{10cm}{2cm}x'\right) = g_{\text{in}}(-5x) = \frac{1}{2}\left[1 + \cos\left(2\pi\frac{x'}{2\mu m}\right)\right]$$

So the field has been laterally demagnified by the imaging system. Notice that lateral demagnification implies angular magnification according to the following diagram:

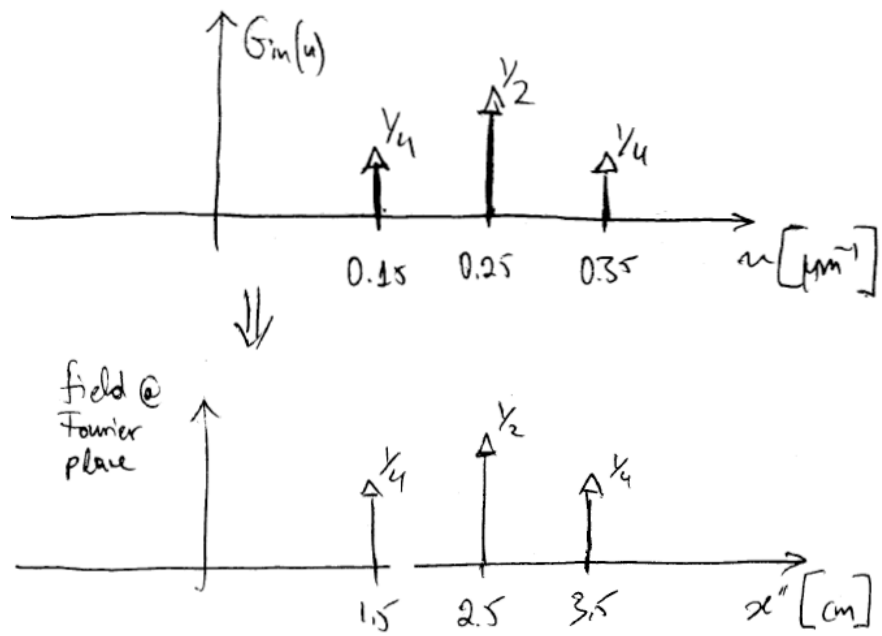


7. Repeat the calculations of problem 6, except this time with illumination of a tilted plane wave incident at angle $\theta = 0.25$ rad with respect to the optical axis.



Solution: This time $g_{\text{in}}(x) = e^{i\frac{2\pi}{\lambda}\sin\theta\cdot x} \cdot t(x)$, where $\sin\theta \approx \theta = 0.25$ (paraxial approximation). Therefore,

$$
\begin{aligned}
g_{in}(x) &= e^{i2\pi\frac{0.25x}{1\mu\text{m}}} \times \frac{1}{2}\left[1 + \cos\left(2\pi\frac{x}{10\mu\text{m}}\right)\right] \\
&= \frac{1}{2}e^{i2\pi\frac{0.25x}{1\mu\text{m}}}\left[1 + \frac{1}{2}e^{i2\pi\frac{x}{10\mu\text{m}}} + \frac{1}{2}e^{-i2\pi\frac{x}{10\mu\text{m}}}\right] \\
&= \frac{1}{2}e^{i2\pi\frac{x}{4\mu\text{m}}} + \frac{1}{4}e^{i2\pi\left(\frac{1}{10}+\frac{1}{4}\right)\frac{x}{\mu\text{m}}} + \frac{1}{4}e^{i2\pi\left(-\frac{1}{10}+\frac{1}{4}\right)\frac{x}{\mu\text{m}}} \\
G_{\text{in}}(u) &= \frac{1}{2}\delta\left(u - \frac{1}{4\mu\text{m}}\right) + \frac{1}{4}\delta(u - 0.35\mu\text{m}^{-1}) + \frac{1}{4}\delta(u - 0.15\mu\text{m}^{-1})
\end{aligned}
$$

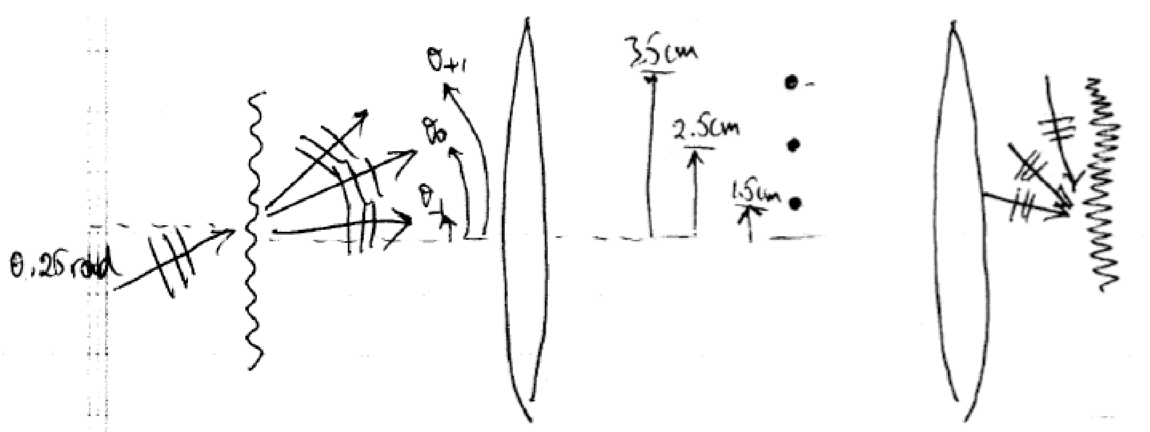


Note 1. We can get this result faster by use of the shift theorem of Fourier transforms:

$$g_{\text{in}}(x) = e^{-i2\pi\frac{x}{4\mu\text{m}}} \cdot t(x) \Rightarrow G_{\text{in}}(u) = T\left(u - \frac{1}{4\mu\text{m}}\right)$$

So it is the result of Problem 6 shifted to the right by $0.25\mu\text{m}^{-1}$.

Note 2. Physical explanation:



The diffracted order angles are:

$$\begin{aligned} \theta_{-1} &\simeq 0.25 - 0.1 &= 0.15 \text{ rad} \\ \theta_{0} &\simeq 0.25 + 0 &= 0.25 \text{ rad} \\ \theta_{+1} &\simeq 0.25 + 0.1 &= 0.35 \text{ rad} \end{aligned}$$

Compare this with the diagram in the answer to Problem 6!

Field at the output plane:

$$g_{\text{out}}(x') = g_{\text{in}}(-5x') = e^{-i2\pi 1.25\frac{x}{\mu m}}\left[1+\cos\left(2\pi\frac{x}{2\mu m}\right)\right]$$

Notice the intensity $|g_{\text{out}}(x')|^2$ is the same as in problem 6. The extra phase factor indicates that the overall output field is propagating "downwards".

8. Repeat problem 7 with a truncated grating of size 1 mm.

Solution: Now the input field is

$$g_{\text{in}} = e^{i2\pi\frac{x}{4\mu\text{m}}} \times \frac{1}{2}\left[1+\cos\left(2\pi\frac{x}{10\mu\text{m}}\right)\right] \times \underbrace{\text{rect}\left(\frac{x}{1\text{mm}}\right)}_{\substack{\text{truncates the grating}\\ \text{to total size of 1 mm}}}$$

We need to compute $G_{in}(u)$. We will do it in two steps:

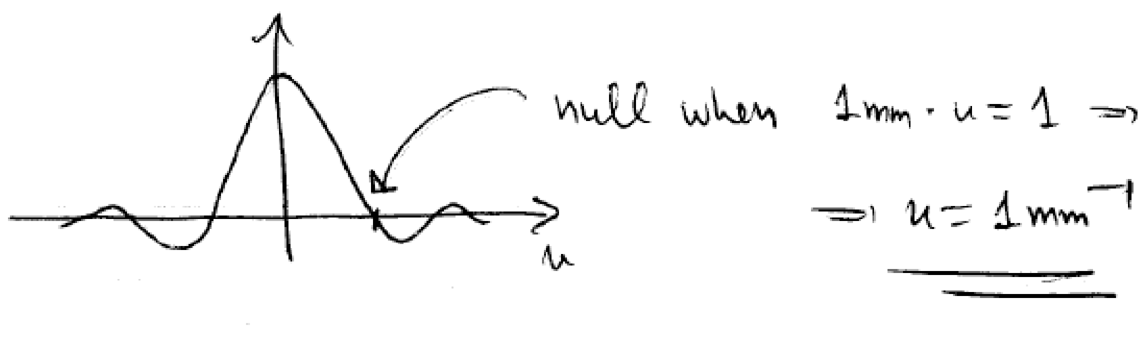
(a) Compute the Fourier transform of $\frac{1}{2}\left[1+\cos\left(2\pi\frac{x}{10\mu\text{m}}\right)\right]\times\text{rect}\left(\frac{x}{1\text{mm}}\right)$ by applying the convolution theorem:

$$f = \frac{1}{2}\left[1+\cos\left(2\pi\frac{x}{10\mu\text{m}}\right)\right] \times \text{rect}\left(\frac{x}{1\text{mm}}\right)$$

$$\mathcal{F} = \left[\frac{1}{2}\delta(u) + \frac{1}{4}\delta\left(u-\frac{1}{10\mu\text{m}}\right) + \frac{1}{4}\delta\left(u+\frac{1}{10\mu\text{m}}\right)\right] \otimes \underbrace{\cancel{(1\text{mm})}}_{\substack{\text{neglect}\\ \text{from now on}}}\ \text{sinc}(1\text{mm}\cdot u)$$

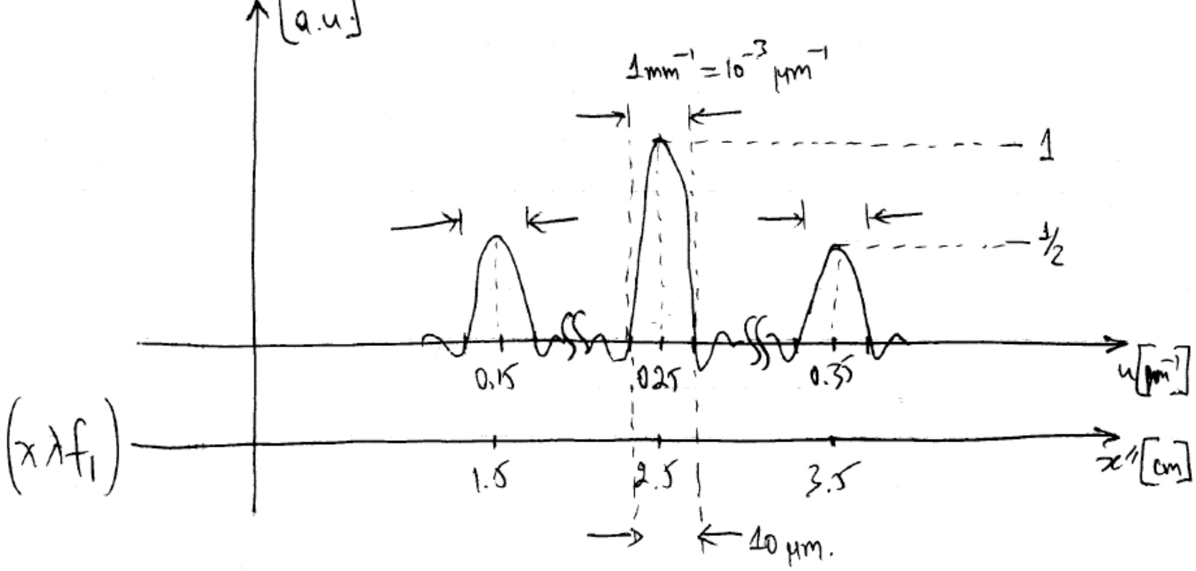
$$= \frac{1}{2}\text{sinc}(1\text{mm}\cdot u) + \frac{1}{4}\text{sinc}\left[1\text{mm}\left(u-\frac{1}{10\mu\text{m}}\right)\right] + \frac{1}{4}\text{sinc}\left[1\text{mm}\left(u+\frac{1}{10\mu\text{m}}\right)\right]$$

Let's plot the first term of this expression before continuing.



(b) Apply the shift theorem to take into account the $e^{i2\pi\frac{x}{4\mu\text{m}}}$ factor (see also problem 7):

$$G_{in}(u) = \tfrac{1}{2}\text{sinc}\left[1\text{mm}\left(u-\tfrac{0.25}{\mu\text{m}}\right)\right] + \tfrac{1}{4}\text{sinc}\left[1\text{mm}\left(u-\tfrac{0.35}{\mu\text{m}}\right)\right] + \tfrac{1}{4}\text{sinc}\left[1\text{mm}\left(u-\tfrac{0.15}{\mu\text{m}}\right)\right]$$

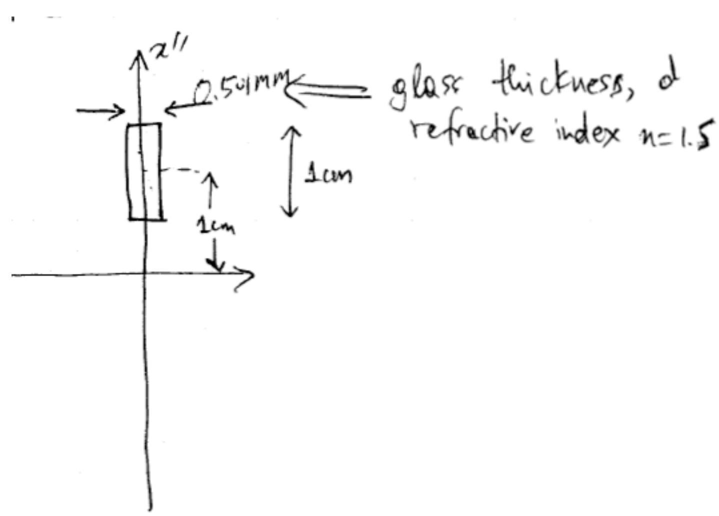


Field at output plane:

$$g_{\text{out}}(x') = g_{\text{in}}(-5x') = e^{-i2\pi 1.25\frac{x}{\mu\text{m}}}\left[1+\cos\left(2\pi\frac{x}{2\mu\text{m}}\right)\right]\text{rect}\left(\frac{x}{0.2\text{mm}}\right)$$

It is still a truncated grating, shrunk by a factor of 5 compared to the original grating.

9. In the optical system of problem 6 (infinitely large grating, on-axis plane wave illumination) we place a small piece of glass at the Fourier plane as follows:



What is the output field? What is the output intensity?

Solution: The piece of glass delays the $+1^{\text{st}}$ order field by a phase equal to:

$$\phi = 2\pi\frac{(n-1)d}{\lambda} = 2\pi\frac{0.5\times 501\mu\text{m}}{1\mu\text{m}} = 501\pi \Rightarrow \phi = \pi \qquad \text{(phase is mod } 2\pi)$$

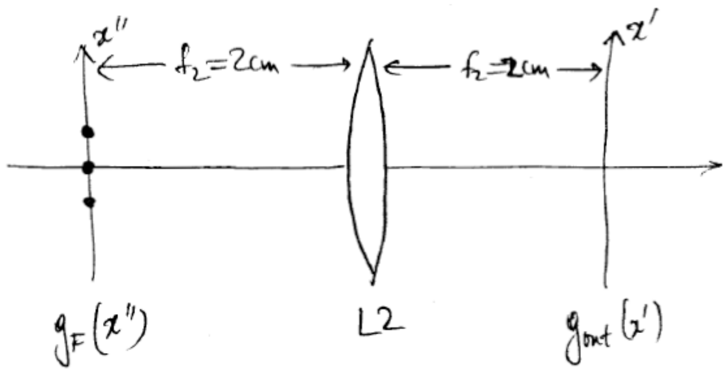
So, immediately after the Fourier plane, the field is:

$$\frac{1}{2}\left[\delta(u)+\frac{1}{2}e^{i\phi}\delta\left(u-\frac{1}{10\mu\mathrm{m}}\right)+\frac{1}{2}\delta\left(u+\frac{1}{10\mu\mathrm{m}}\right)\right]=\frac{1}{2}\left[\delta(u)-\frac{1}{2}\delta\left(u-\frac{1}{10\mu\mathrm{m}}\right)+\frac{1}{2}\delta\left(u+\frac{1}{10\mu\mathrm{m}}\right)\right]$$

We simplify the phase delay, $e^{i\phi}=e^{i\pi}=-1$. Next, switch to coordinates $x''=\lambda f_1 u$:

$$\frac{1}{2}\left[\delta(x'')-\frac{1}{2}\delta\left(x''-1\mathrm{cm}\right)+\frac{1}{2}\delta\left(x''+1\mathrm{cm}\right)\right]\equiv g_F(x'')$$

Now consider the second half of the 4F system:



Since L2 is acting as a Fourier-transforming lens,

$$\begin{aligned}
g_{\text{out}}(x') = G_F\left(\frac{x''}{\lambda f_2}\right) &= \frac{1}{2}\left[1-\frac{1}{2}e^{i2\pi\frac{x'}{2\mu\mathrm{m}}}+\frac{1}{2}e^{-i2\pi\frac{x'}{2\mu\mathrm{m}}}\right] \\
&= \frac{1}{2}\left[1-i\sin\left(2\pi\frac{x'}{2\mu\mathrm{m}}\right)\right] \quad \rightarrow \quad \text{field} \\
|g_{\text{out}}(x')|^2 &= \frac{1}{2}\left[1+\sin^2\left(2\pi\frac{x'}{2\mu\mathrm{m}}\right)\right] \quad \rightarrow \quad \text{intensity}
\end{aligned}$$

10. Consider the 4F optical system shown in Figure B, where lenses L1, L2 are identical with focal length f. A thin transparency with arbitrary transmission function $t(x)$ is placed at the input plane of the system, and illuminated with a monochromatic, coherent plane wave at wavelength λ, incident on-axis. At the Fourier plane of the system we place the amplitude filter shown in Figure C. The filter is opaque everywhere except over two thin stripes of width a, located symmetrically around the y'' axis. The distance between the stripe centers is $x_0 > a$.

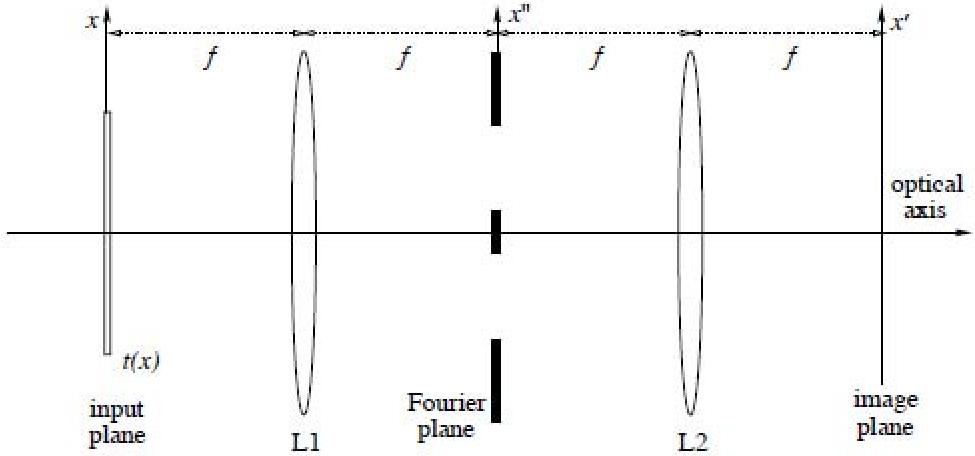


Figure B

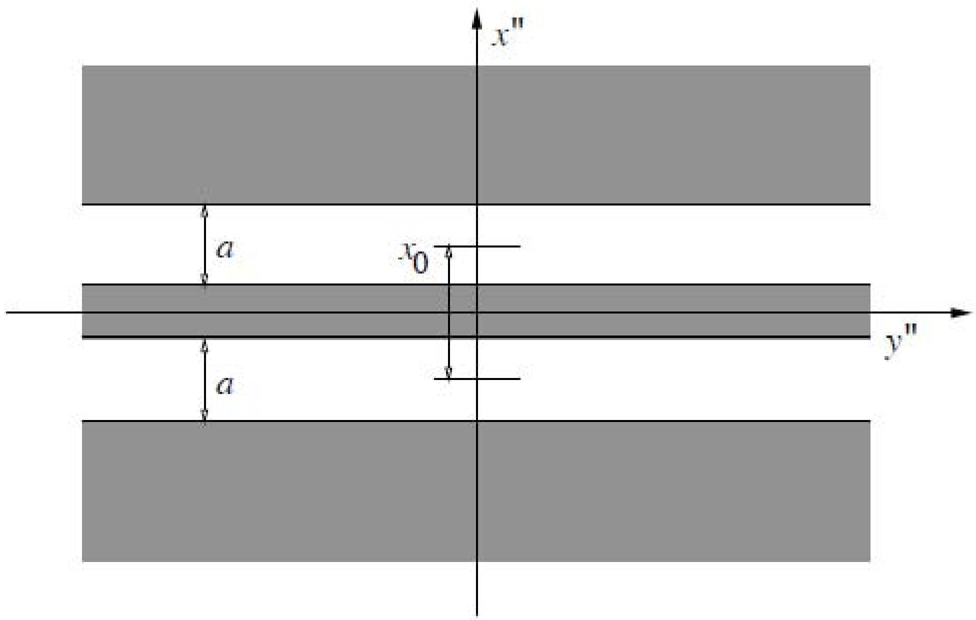


Figure C

(a) Which range of spatial frequencies must $t(x)$ contain for the system to transmit any light to its image plane?

Solution: The system admits frequencies

$$\frac{\frac{x_0}{2} - \frac{a}{2}}{\lambda f} \leq u \leq \frac{\frac{x_0}{2} + \frac{a}{2}}{\lambda f}$$

$$-\left(\frac{\frac{x_0}{2} + \frac{a}{2}}{\lambda f}\right) \leq u \leq -\left(\frac{\frac{x_0}{2} - \frac{a}{2}}{\lambda f}\right)$$

(b) Write an expression for the field at the image plane as the convolution of $t(x)$ with the coherent impulse response of this system.

Solution:

$$
\begin{aligned}
H(u) &= \mathrm{rect}\left(\frac{u - \frac{x_0}{2\lambda f}}{\frac{a}{\lambda f}}\right) + \mathrm{rect}\left(\frac{u + \frac{x_0}{2\lambda f}}{\frac{a}{\lambda f}}\right) \\
h(x) &= \frac{a}{\lambda f}\mathrm{sinc}\left(\frac{ax'}{\lambda f}\right) e^{-i2\pi\frac{x_0 x'}{2\lambda f}} + \frac{a}{\lambda f}\mathrm{sinc}\left(\frac{ax'}{\lambda f}\right) e^{i2\pi\frac{x_0 x'}{2\lambda f}} \\
&= \frac{2a}{\lambda f}\mathrm{sinc}\left(\frac{ax'}{\lambda f}\right)\cos\left(\frac{\pi x_0 x'}{\lambda f}\right)
\end{aligned}
$$

The output is:

$$
g(x') = \frac{2a}{\lambda f}\int_{-\infty}^{\infty} t(x'-x)\mathrm{sinc}\left(\frac{ax}{\lambda f}\right)\cos\left(\frac{\pi x_0 x}{\lambda f}\right) dx = t(x)\otimes h(x)\Big|_{x'}
$$