# APPENDIX

## Further Material on Panel Methods and Strip Theory

# Panel Methods

## 13.024 Numerical Marine Hydrodynamics

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During the last decade, advancements in computer technology have made possible the development of new classes of three-dimensional numerical tools for analyzing problems in Naval Architecture, such as ship wave resistance and motions.

Early attempts to model ships in potential flow focused on variations of slender body and strip theory to study simplified body geometries and free surface conditions.

As computing power increased, so did the development of three-dimensional methods. Of these, considerable attention has been received by boundary element or panel methods.

## Panel Methods at a Glance

- Distribute sources and dipoles on body
- Discretize
- Green's Theorem gives system of equations for singularity strength on each panel in terms of boundary conditions
- Forces on body found from flow solution



Panel methods attempt to solve the Laplace equation in the fluid domain by distributing sources and dipoles on the body and, in some methods, on the free surface.

These surfaces are divided into panels, each one associated with a source and dipole distribution of unknown strength.

Green's theorem relates the source and dipole distribution strength to the potential and normal velocity on each panel.

The boundary conditions to be applied to the problem are often linearized and they determine either the potential or the normal velocity on each panel.

Having solved for the unknown source and dipole strengths, Green's theorem may be used to find the potential at any point in the fluid domain.

Hydrodynamic forces are found from pressure integration and are used with Newton's Law to determine motions.

## Cases to be Examined

- Unbounded fluid flows
  - Steady motion through unsteady flow field
- Lifting flows
  - Forced motion in free stream
- Wave flows
  - Ship under steady motion in calm water
  - Free motions of buoy in waves

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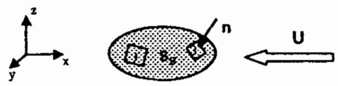
Panel methods can ultimately solve complex problems involving free motions of forward moving vessels with lifting surfaces in incident waves. Jumping right in to the formulation of such a problem, however, would be rather overwhelming.

We will, therefore, start by formulating a simple problem and move on to progressively more complex cases.

# Flow in Unbounded Fluid

## Non-lifting body in unsteady flow




Unsteady incident

Total potential: $\Phi = \varphi + \phi - Ux$

Perturbation

Free stream

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Body advancing with speed U in an unsteady flow field

Body-fixed coordinate system.

Unit normal n, to body surface $S_B$, pointing out of fluid.

Reasons to solve this problem:

- Get pressure distribution on body
- Determine added mass
- Introduce techniques to be used with more complex problems

Separate total potential into:

- free stream potential
- incident potential excluding free stream
- perturbation from incident potential

# Body Boundary Condition

(No flux condition)

$$\frac{\partial \Phi}{\partial n} = 0 \Rightarrow$$

$$\Rightarrow \frac{\partial \varphi}{\partial n} = \left(U\hat{i} - \nabla \phi\right)\cdot \vec{n}$$

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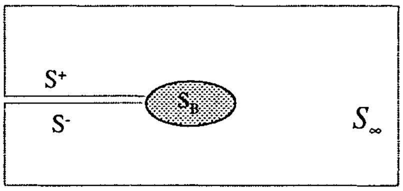
The normal velocity of fluid must be zero in the body-fixed coordinate system.

Using the decomposition of total potential into its components, it follows that the normal velocity of the perturbation flow must be equal and opposite to the the incident flow, which is given.

# Boundary Integral Equation

## Green's Theorem for field points on body surface, $S_B$




$$\varphi = \frac{1}{2\pi} \iint_{S_B} \left[ G \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial G}{\partial n} \right] dS$$

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The application of Green's second identity transforms the boundary value problem stated previously into a boundary integral equation. This facilitates the numerical solution as the entire fluid volume does not have to be discretized.

The integral over the part of the control surface at infinity vanishes because the flow sufficiently far from the body is undisturbed. The integrals over the connecting surfaces $S^+$ and $S^-$ cancel each other out. So what is left defines the potential on the body in terms of a source (G) and dipole (dG/dn) distribution on the body surface. The strength of the source distribution is given by the magnitude of the normal velocity on the body, while the strength of the dipole distribution is equal to the magnitude of the potential on the body.

For this particular problem, $d\varphi/dn$ is given from the boundary condition, while $\varphi$ is unknown.

# Numerical Solution

## Discretize integral equation and substitute body BC

$$2\pi\varphi_i + \sum_j \varphi_j \frac{\partial G_{ij}}{\partial n_j} = \sum_j G_{ij}\left( U n_{jx} - \frac{\partial \phi_j}{\partial n_j} \right)$$

System of linear equations for unknown $\varphi_i$

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The body is discretized into $n$ panels, each of area $A_j$.

The singularity distribution on each panel can be constant, or of higher order. In any case, $G_{ij}$ is the potential at the control point of panel $i$, due the source distribution on panel $j$.

Having a higher order distribution on each panel results in less panels needed for convergence and leads to a more robust way of calculating the tangential velocities on the body, if needed. More on higher order distributions later.

In the above system of $n$ linear equations, the RHS is known from the body boundary condition. The potential on each panel may thus be found by a standard linear solver. The flow is hence completely specified.

## Hydrodynamic Forces and Moments

$$\vec{F} = -\rho \sum_i \left( \frac{\partial \Phi_i}{\partial t} + \frac{1}{2} \nabla \Phi_i \cdot \nabla \Phi_i \right) \vec{n}_i A_i$$

$$\vec{M} = -\rho \sum_i \left( \frac{\partial \Phi_i}{\partial t} + \frac{1}{2} \nabla \Phi_i \cdot \nabla \Phi_i \right) (\vec{x}_i \times \vec{n}_i) A_i$$

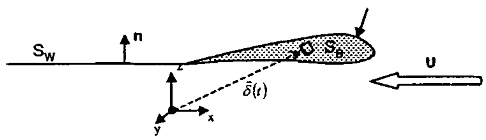


From the potential and its gradient on the body, it is straightforward to determine the pressure distribution and hence the hydrodynamic forces and moments.

These forces are often linearized by assuming small perturbations about the free stream.

# Lifting Flows

## Forced motions in steady free stream

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Lifting surfaces common in Naval Architecture include hydrofoils, rudders, control fins, sailboat keels and sails, and catamaran hulls.

A special treatment is needed for such lift-producing bodies because the potential flow solution to the problem as previously formulated would include infinite velocities at the sharp trailing edge of the foil under angle of attack.

In order to ensure a smooth flow at the trailing edge, which in real life is attained due to the presence of viscosity, the wake shed from the hydrofoil must be modeled.

The problem examined here involves a hydrofoil performing small motions about a steady forward motion. So in addition to the effect of lift, there is a new element in the formulation of the boundary value problem. The body now moves with respect to the coordinate system, which is translating with a steady velocity, U.

# Wake Model

Thin free vortex sheet

Across wake:

- Continuous normal velocity
- Discontinuous potential (jump=$\Delta\Phi$)
- Zero pressure jump:

$$\frac{\partial \Delta\varphi}{\partial t} - U\frac{\partial \Delta\varphi}{\partial x} = 0$$

(Bernoulli, linearized about free stream)

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The wake is modeled as a free vortex sheet shed downstream from the trailing edge of the foil. Mathematically this may be defined as a dipole distribution on a surface $S_W$, of zero thickness.

The operator $\Delta$ denotes a jump in a quantity from one side of the wake to the other.

Across the wake we have continuity of normal velocity and a jump in potential. The pressure on both sides of the wake should be equal because otherwise we would have infinite particle acceleration since the wake is infinitesimally thin.

The wake can be shed either straight back, following the free stream, or it could have each point follow the total velocity induced at its location by both the foil and the rest of the wake. In general, however, the additional computational load and stability problems do not justify the slight increase in accuracy achieved by tracking the exact position of the wake.

# Kutta Condition

- Requires zero pressure jump at trailing edge of foil.
- This is ensured by continuity of potential, together with the condition of zero pressure jump across the wake

$$\Phi_{TE:body} = \Phi_{TE:wake}$$



The flow past a lifting body cannot be uniquely determined unless some additional condition is specified which sets the amount of circulation produced by the foil. As previously formulated, (without the wake) there would be no circulation and the velocity at the sharp trailing edge of the foil would be infinite.

This situation may be avoided if a Kutta condition requiring tangential velocities at the trailing edge is enforced. An alternate way of enforcing this condition is to require continuous pressure at the trailing edge. In fact, this condition is preferred here because it can be easily linearized about the free stream.

The requirement of zero pressure jump at the trailing edge in the wake is already satisfied as we saw before. Thus, by also requiring continuity of potential from the body into the wake at the trailing edge, the Kutta condition of zero pressure jump on the body is automatically satisfied.

# Forced Motions

Displacement about frame of reference due to translation and rotation:

$$\vec{\delta}(\vec{x},t)=\vec{\xi}_T(t)+\vec{\xi}_R(t)\times\vec{x}$$

$$\vec{\xi}_T=(\xi_1,\xi_2,\xi_3) \qquad \vec{\xi}_R=(\xi_4,\xi_5,\xi_6)$$

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Since for this problem the body is not fixed with respect to the coordinate system, we need to define its motions.

The rigid body motions in six degree of freedom can be fully described by a translation and a rotation vector.

The displacement of any point on the body with respect to its original position may be described in terms of these two vectors and its original displacement from the origin.

# Body Boundary Conditions

Applied at exact body surface:

$$\frac{\partial\varphi}{\partial n} = \left( U\hat{i} + \frac{\partial\vec{\delta}}{\partial t} \right) \cdot \vec{n}$$

Linearized and applied at mean position of body (assuming small body motions)

$$\frac{\partial\varphi}{\partial n} = \frac{d\vec{\xi}_T}{dt} \cdot \vec{n} + \frac{d\vec{\xi}_R}{dt} \cdot (\vec{x} \times \vec{n}) + U(\xi_5 n_z - \xi_6 n_y + n_x)$$

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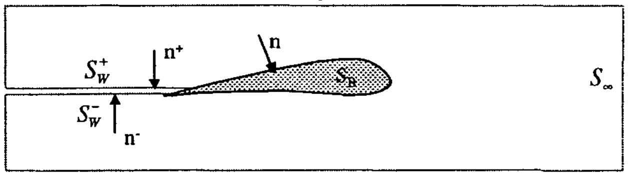
The boundary condition is defined in terms of the body motions in order to preserve the zero normal flux requirement.

This condition should, strictly, be applied to the exact position of the body surface. This would require the re-discretization of the body surface at each time step. Although some panel methods take this approach to the problem, the solution becomes much easier numerically if the motions can be assumed small and can be linearized about the mean position of the body.

The linear body boundary condition is derived by applying a Taylor expansion about the mean body position and retaining linear terms.

## Boundary Integral Equation

Green's Theorem for field points on the body




$$2\pi\varphi + \iint_{S_B} \varphi \frac{\partial G}{\partial n} dS + \iint_{S_W} \Delta\varphi \frac{\partial G}{\partial n} dS = \iint_{S_B} G \frac{\partial \varphi}{\partial n} dS$$

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Green's theorem is used once more to derive the integral equations.

As with the non-lifting case, the integral over the control surface at infinity vanishes. The integral over the connecting surfaces that run over the wake do not completely cancel each other, however, due to the discontinuity in potential across the wake. Instead, we get a term involving an integral over $S_W$ of the potential jump multiplied by the dipole potential.

The problem can no longer be solved by placing panels only on the body. The wake also needs to be discretized.

# Numerical Solution

## Discretize and Fourier Transform integral equation, using body BC

$$2\pi\varphi_k + \sum_{j=1}^{N_{body}} \varphi_j \frac{\partial G_{kj}}{\partial n_j} + \sum_{j=N_{body}+1}^{N_{wake}} \Delta\varphi_j \frac{\partial G_{kj}}{\partial n_j} =$$

$$= \sum_{j=1}^{N_{body}} G_{kj} \left\{ i\omega \left[ \vec{\xi}_T \cdot \vec{n}_j + \vec{\xi}_R \cdot \left( \vec{x}_j \times \vec{n}_j \right) \right] + U\left( \xi_5 n_{zj} - \xi_6 n_{yj} \right) \right\}$$

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Due to the wake shed downstream, this problem has memory and thus the solution depends on the flow at previous time instances. This means that the solution has to be evolved in time, or needs to be solved by tanking the Fourier transform and solving for each frequency component present.

The integral equation shown above is in the frequency domain, if the forced motions are sinusoidal. Solution in the time domain would require the numerical evaluation of the time derivatives.

The system of equations shown above are simply the integral equation at each panel, with the body boundary condition substituted at the RHS.

There are, however, more unknowns than integral equations due to the extra panels of unknown potential jump in the wake. The extra equations to close the problem are derived from the wake condition of zero pressure jump.

# Numerical Solution

## Wake Condition:

To be solved simultaneously with integral equation

at T.E.... $\Delta\varphi_{wake} = \left[\varphi_{upper} - \varphi_{lower}\right]_{body}$

in wake... $i\omega\,\Delta\varphi_k - U\dfrac{\partial\Delta\varphi_k}{\partial x} = 0$

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For a solution in the frequency domain, the wake condition is discretized and yields one extra equation for each panel in the wake.

If the problem is solved in the time domain, the potential jump can be expressed exclusively in terms of the potential jump at panels during the previous time step. Instead of unknown potential jump on the entire set of wake panels, the only extra unknowns would thus be a strip of panels immediately downstream of the trailing edge of the foil. This reduces the size of the matrix to be solved, at the cost of having to evolve the solution in time.

The wake condition involves the evaluation of spatial derivatives of the potential jump. We will examine methods for doing this later.

# Hydrodynamic Forces and Moments

$$\vec{F} = -\rho \sum_{i=1}^{N_{body}} \left( i\omega \varphi_i - U \frac{\partial \varphi_i}{\partial x} + \frac{1}{2} \nabla \varphi_i \cdot \nabla \varphi_i \right) \vec{n}_i A_i$$

$$\vec{M} = -\rho \sum_{i=1}^{N_{body}} \left( i\omega \varphi_i - U \frac{\partial \varphi_i}{\partial x} + \frac{1}{2} \nabla \varphi_i \cdot \nabla \varphi_i \right) (\vec{x}_i \times \vec{n}_i) A_i$$



After having solved the flow, the calculation of the hydrodynamic forces and moments is again a matter of integrating the pressure distribution over the surface of the body.

For steady flows it is also possible to determine the lift and drag based on a Trefftz plane integration:

$$D = \frac{1}{2} \rho \int_{-s/2}^{s/2} \Delta\Phi \frac{\partial \Phi}{\partial z} dy$$

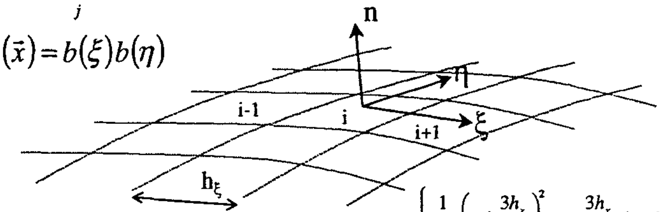
$$L = \rho U \int_{-s/2}^{s/2} \Delta\Phi \, dy$$

where $s$ is the span of the foil.

# Higher Order Potential Distribution

$$\varphi(\vec{x}) = \sum_j c_j B_j(\vec{x})$$

$$B(\vec{x}) = b(\xi)b(\eta)$$




Bi-quadratic B-Spline: $$b(x) = \begin{cases} \dfrac{1}{2h_x^2}\left(x + \dfrac{3h_x}{2}\right)^2, & -\dfrac{3h_x}{2} < x < -\dfrac{h_x}{2} \\ \dfrac{1}{h_x^2}\left(-x^2 + \dfrac{3h_x^2}{4}\right) & -\dfrac{h_x}{2} < x < \dfrac{h_x}{2} \\ \dfrac{1}{2h_x^2}\left(-x + \dfrac{3h_x}{2}\right)^2, & \dfrac{h_x}{2} < x < \dfrac{3h_x}{2} \end{cases}$$

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In general, the potential distribution on each panel is not constant.

A B-spline representation, for example, represents the potential as a summation with weight $c_j$ of all basis functions $B_j$ centered on each panel *j*. A second order basis function is shown above. Note that the field point needs to be converted into local panel coordinates for the evaluation of the basis functions.

The spline coefficients $c_j$, determine the amount of contribution from each panel, and become the unknowns in the integral equations. Due to the overlap of the basis functions in determining the potential at the center of several panels, however, the unknown spline coefficients are still one per panel.

A consequence of higher order B-spline distributions is that end conditions need to be specified at the edges of the spline sheets, so that the spline coefficients may be uniquely determined.

Higher order singularity distributions require fewer panels to achieve numerical flow convergence. Note that similarly, geometrical convergence may be achieved faster if the surface is described not in terms of flat quadrilateral panels, but by B-spline surfaces.

# Evaluation of tangential derivatives

constant distribution: $$\frac{\partial \varphi_i}{\partial \xi} = \frac{\varphi_{i+1} - \varphi_{i-1}}{h_{i\xi}}$$

higher order: $$\frac{\partial \varphi}{\partial \xi} = \sum_j c_j \frac{\partial b_j(\xi)}{\partial \xi} b_j(\eta)$$



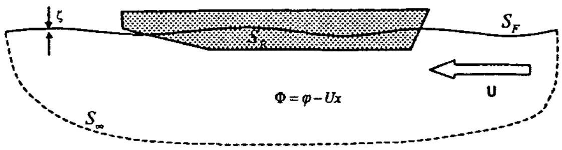
As already seen, there is a need for the calculation of the tangential velocity on a panel. This may be done by finite differences, and one example is shown above. Of course, any other finite differences scheme could be used, provided that it does not make the overall method unstable.

If the potential distribution is of higher order then the tangential velocities can be found analytically from direct differentiation of the basis functions.

Note that the above derivatives are given in panel local coordinates. Since the derivatives are usually required with respect to the global coordinate system, a transformation is needed for each panel.

# Free Surface Flows

## Steady ship motion in calm water




Total potential: $\Phi = \varphi - Ux$

Perturbation    Free stream

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The evaluation of the steady wave resistance of a ship has always been of great importance to Naval Architects. Three dimensional panel methods have the ability to estimate this quantity without resorting to expensive towing tank testing.

We will formulate the problem of a ship advancing steadily through calm water, linearizing the solution about the free stream. The total flow is therefore broken down into a free stream and a small perturbation flow components.

# Free Surface Boundary Conditions

Dynamic:
zero total pressure on free surface

$$p_a = -\rho\left(\frac{\partial\varphi}{\partial t} - U\frac{\partial\varphi}{\partial x} + \frac{1}{2}\nabla\varphi\cdot\nabla\varphi + g\zeta\right) = 0$$

Kinematic:
particle on free surface remains there

$$\left(\frac{\partial}{\partial t} + \nabla\Phi\cdot\nabla\right)[z-\zeta] = 0$$

$\zeta$ = wave elevation

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Boundary conditions are required to determine the behavior of the flow near the free surface, and hence uniquely determine the solution.

The dynamic condition requires the pressure at the free surface to be equal to the atmospheric pressure, which will be taken arbitrarily to be equal to zero. The condition is thus expressed by the Bernoulli equation above.

The kinematic condition requires that a particle on the free surface remains on the free surface forever. This means that the material derivative of its vertical distance from the free surface should be zero.

# Linearization about Free Stream

## Kelvin boundary conditions

$$\frac{\partial \phi}{\partial t} - U\frac{\partial \phi}{\partial x} = -g\zeta$$

$$\frac{\partial \zeta}{\partial t} - U\frac{\partial \zeta}{\partial x} = \frac{\partial \phi}{\partial z}$$

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The free surface boundary conditions previously stated are non-linear and are to be applied at the exact position of the free surface, which is unknown. The numerical solution algorithm becomes much simpler and computationally efficient if these conditions can be linearized and applied to a known surface.

The above linear conditions, also known as the Kelvin free surface boundary conditions, were derived using a Taylor expansion about z=0 for small wave elevations and slopes, and ignoring higher order terms.

The wave elevation can, of course, be eliminated by combining the two equations, resulting in a condition involving only the perturbation potential and its temporal and spatial derivatives.

## Boundary Integral Equation

$$2\pi\varphi = \iint_{S_B+S_F}\left[G\frac{\partial\varphi}{\partial n}-\varphi\frac{\partial G}{\partial n}\right]dS$$

Take Kelvin wave source as the Green function
As shown in hydrodynamics review of this course,
waterline integral replaces integral over free surface:

$$2\pi\varphi = \iint_{S_B}\left[G\frac{\partial\varphi}{\partial n}-\varphi\frac{\partial G}{\partial n}\right]dS+\frac{U^2}{g}\oint_{wl}\left[G\frac{\partial\varphi}{\partial x}-\varphi\frac{\partial G}{\partial x}\right]\frac{n_x}{\cos\gamma}dl$$

$\gamma$ = flare angle

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Green's theorem is used again to derive the integral equation, only this time an integral over the free surface is needed, as well as over the body. Using a Green function that satisfies the linear free surface conditions, however, the free surface integral may be collapsed into a waterline integral. This means that no panels are needed on the free surface.

One difficulty is that the first derivative of the very complicated Green function is required, but this can be done numerically.

Note that this formulation of the integral equation relies on the use of the linearized Kelvin boundary conditions. This is because Green functions satisfying any other free surface linearization are not readily obtainable.

# Numerical Solution

## Discretize Integral Equation and Substitute Body BC

$$2\pi\varphi_k + \sum_j \varphi_j \frac{\partial G_{kj}}{\partial n_j} + \frac{U^2}{g}\sum_{j\in WL}\varphi_j \frac{\partial G_{kj}}{\partial x}\frac{n_j}{h_j\cos\gamma_j} =$$

$$= U\sum_j G_{kj} n_{jx} + \frac{U^3}{g}\sum_{j\in WL} G_{kj}\frac{n_{jx}^2}{h_j\cos\gamma_j}$$

Linear system of equations for $\varphi_k$



Discretizing the integral equation, a system of linear equations is obtained for the potential on each panel, as before.

Care must be taken in evaluating the waterline integral, since the value of the potential on the free surface needs to be estimated from the potential on the body, which is often discretized only below the z=0 plane.

# Wave Resistance

## From Momentum Conservation

$$R_W = \iint_{S_B} p\, n_x\, dS - \frac{\rho g}{2} \oint_{WL} \zeta^2 \frac{n_x}{\cos\gamma} dl$$

(Proof follows)

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The wave resistance of a ship may be found from the above formula after the flow has been solved.

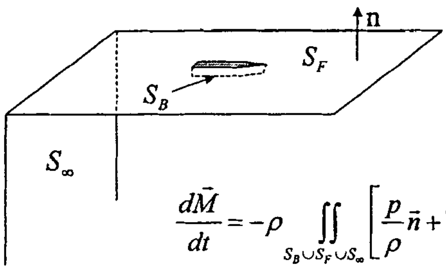
The pressure should *include* the quadratic term in Bernoulli's equation, even though terms of comparable magnitude have been omitted in the linearization of the free surface boundary conditions.

If the quadratic terms are omitted, the resistance of full-shaped vessels is over-predicted. The reason for this is that close to such bluff bodies, which is where we are performing the pressure integration, the perturbation potential is actually of the same order as the free stream potential, so the linearization is not accurate.

Linearizing about a double-body basis flow, as we will see later, solves this problem and the quadratic terms are not as important.

# Wave Resistance

## Conservation of Momentum




$$\frac{d\bar{M}}{dt} = -\rho \iint_{S_B \cup S_F \cup S_\infty} \left[ \frac{p}{\rho}\vec{n} + \nabla\Phi\left(\nabla\Phi\cdot\vec{n}\right) \right] dS = 0$$

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The proof of the formula given for the wave resistance follows from the application of the momentum conservation principle inside an appropriately chosen control volume of fluid.

Within the enclosed volume, the rate of change of fluid momentum vanishes.

Note that because of the radiation condition, the only surface at infinity where the integrand does not vanish is far downstream of the body, at $S_\infty$.

# Control Volume at Exact Position of Fluid Surfaces

Using boundary conditions:

$$R_W = \iint_{S_B} p\, n_x\, dS = -\rho \iint_{S_\infty} \left[ \frac{p}{\rho} n_x + \frac{\partial \Phi}{\partial x} \frac{\partial \Phi}{\partial n} \right] dS$$

From radiation condition and Bernoulli:

$$R_W = -\frac{\rho g}{2} \int_{C_d} \zeta^2 dy - \frac{\rho}{2} \iint_{S_d} \left[ -\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2 \right] dS$$

$C_d$ is the intersection of $S_\infty$ with z=0
$S_d$ is the part of $S_\infty$ lying below z=0

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Taking the exact wetted surface of the hull, the exact position of the free surface, and $S_\infty$, as the control surfaces (all at rest with respect to the body) and using the body and free surface boundary conditions, the only terms that do not vanish are the pressure integration over the body wetted surface (which is defined as the non-linear wave resistance), and the momentum flux and pressure integration at infinity.

The fluid velocity in the x and z directions may be found from the Kelvin free surface boundary conditions, and the fluid pressure from Bernoulli.

The resulting expression is an exact representation of the wave resistance in terms of far-field quantities.

# Control volume at linearized position of fluid surfaces

Using Kelvin boundary conditions and Bernoulli:

$$\iint_{\bar{S}_B} p\, n_x\, dS - \rho g \iint_{\bar{S}_F} \zeta \frac{\partial \zeta}{\partial x}\, dS + \frac{\rho}{2} \iint_{S_d} \left[ -\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2 \right] dS = 0$$

After application of Stokes theorem:

$$\iint_{\bar{S}_B} p\, n_x\, dS - \frac{\rho g}{2} \oint_{WL} \zeta^2 \frac{n_x}{\cos \gamma}\, dl = -\frac{\rho g}{2} \int_{C_d} \zeta^2 dy - \frac{\rho}{2} \iint_{S_d} \left[ -\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2 \right] dS$$

RHS is equal to wave resistance, $R_W$, from momentum conservation in control volume bound by exact surfaces



Repeating the same procedure for a control volume bound by the linearized free surface (z=0 plane), the body below z=0, and the same surface at infinity, a similar expression is derived. This time, the momentum flux across the free surface does not vanish because the normal fluid velocity at z=0 is not zero. The Kelvin conditions are used to express the fluid velocities on the free surface in terms of the wave elevation.

Finally, an application of Stokes theorem transforms the surface integral over the z=0 plane to a pair of line integrals at the body and at infinity.

The line integral along with the surface integral at infinity are recognized as the wave resistance as previously derived using a control volume bound by the exact free surface.

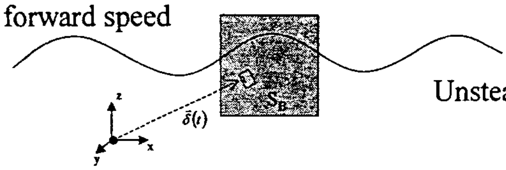
An expression is therefore derived for wave resistance in terms of near-field quantities, starting from the principle of momentum conservation. Comparing this expression to the one derived from pressure integration, we observe that they are similar, but the waterline integral terms have the opposite sign!!

This paradox is due to the inconsistency of retaining second order terms in the definition of wave resistance, but omitting them from the free surface linearization, as previously mentioned. As the beam of the ship approaches zero, the waterline integral term vanishes and the two definitions are in agreement.

# Free Surface Flows

## Free Motions of Body in Waves

Zero forward speed



Unsteady incident

Total potential: $\Phi = \varphi + \phi$

Perturbation

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The determination of the motions of floating bodies in waves is another problem of interest to ocean engineers. Here we will examine the panel method solution of a buoy in incident monochromatic waves. Solutions to more complex problems with forward speed and multiple frequencies can be easily obtained by a simple extension of this problem and the previous one examined.

Since there is no forward speed in this problem, the total potential is divided into the incident wave and perturbation potentials.

## Solution Method

System of Equations:

- Boundary Integral Equation
- Body Boundary Condition
- Equations of Motion

Unknowns:

- Potential on each panel
- Normal Velocity on each panel
- Body motions

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So far the motions of the body have been prescribed, which resulted in the body boundary conditions being completely specified. For freely floating bodies in waves, however, the body boundary condition is a function of the motions, which are unknown. The motions are connected to the hydrodynamic forces through the equations of motion to close the problem.

## System of Equations

Boundary integral equation

$$2\pi\varphi = \iint_{S_B}\left[G\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial G}{\partial n}\right]dS + \frac{U^2}{g}\oint_{wl}\left[G\frac{\partial\varphi}{\partial x} - \varphi\frac{\partial G}{\partial x}\right]\frac{n_x}{\cos\gamma}dl$$

Body boundary condition

$$\frac{\partial\varphi}{\partial n} = \left(\frac{\partial\vec{\delta}}{\partial t} - \nabla\phi\right)\cdot\vec{n}$$



As before, the body boundary condition may be substituted into the integral equation. This time, however, the body motions are unknown, so the integral needs to be solved simultaneously with the equations of motion.

# Equations of Motion

$$M\ddot{\vec{\xi}} + C\vec{\xi} = -\rho \iint_{S_B} \left[ \frac{\partial \Phi}{\partial t} + \frac{1}{2} \nabla\Phi \cdot \nabla\Phi \right] \vec{n}\, dS$$

$$\vec{\delta} = \vec{\xi}_T + \vec{\xi}_R \times \vec{x}$$
$$\vec{\xi}_T = (\xi_1, \xi_2, \xi_3)$$
$$\vec{\xi}_R = (\xi_4, \xi_5, \xi_6)$$
$$\vec{\xi} = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)$$

$$(n_1, n_2, n_3) = \vec{n}$$
$$(n_4, n_5, n_6) = \vec{x} \times \vec{n}$$
$$M = \textit{inertia matrix}$$
$$C = \textit{matrix of restoring coefs}$$

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The equations of motion balance the inertia forces and the hydrostatic restoring forces with the hydrodynamic forces obtained from the flow solution.

A notation is adopted that merges the translation and rotation vectors so that the equations of motion become a six-dimensional matrix equation, balancing both forces and moments.

# Rankine Panel Methods

- Panels both on body and free surface
- Boundary integral equation becomes:

$$2\pi\varphi = \iint_{S_B+S_F} \left[ G\frac{\partial \varphi}{\partial n} - \varphi \frac{\partial G}{\partial n} \right] dS$$

- Use Rankine source (G=1/4πr) as elementary singularity
- Boundary conditions determine potential and normal velocity on free surface
- Linearize about basis flow (double body)



Rankine panel methods distribute panels on both the body and the free surface. They thus have a greater freedom in the free surface boundary conditions that they can apply. This comes at the expense of introducing extra errors due to the discretization of the free surface.

The integral equation retains the free surface term (without collapsing it into a waterline integral as with Neumann-Kelvin methods) and thus has extra unknown source and dipole distributions associated with the free surface panels. These are found from the dynamic and kinematic free surface boundary conditions.

Another advantage of Rankine panel methods is that they do not have to have their solution linearized about the free stream, which is rather poor especially near the ends of the vessel. Instead, they can linearize the solution about a double-body basis flow, which produces more accurate results.

# Discretization Issues

- Distortion of the free surface
  - Dispersion
  - Damping
- Stability
  - Spatial
  - Temporal
- Radiation condition
  - Truncation errors and domain size sensitivity

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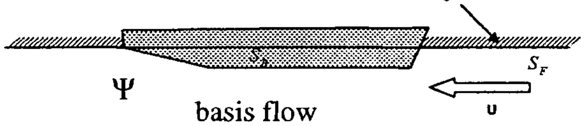
A discrete free surface has a different dispersion relation than a continuous one.. For finite panel sizes spurious wavelengths smaller than five panel lengths are supported and need to be filtered out. Any damping of the numerical method (i.e. Rayleigh viscosity), so that the radiation condition may be satisfied, also affects the numerical dispersion relation.

For a convergent numerical algorithm the numerical dispersion relation should approach the continuous dispersion relation in the limit of infinitesimally small panel sizes. The numerical dispersion relation results in stability criteria, from which required relations between quantities such as panel dimensions, Froude number, time step, can be derived.

Another difference between the continuous and numerical free surfaces is the truncation of the free surface. The condition at the edge of the computational domain should be such that the sensitivity of the solution to the size of the domain is minimized. One way of imposing the radiation condition so that reflected waves from the edge of the domain are minimized is to apply matching at some control volume around the fluid domain which contains a flow satisfying the radiation condition. An alternate (easier) way is to use a numerical beach where the kinematic boundary condition is modified to allow a mass flux through the free surface (Newtonian cooling), thus damping wavelengths less than about twice the extent of the beach.

## Linearization about basis flow

$$\frac{\partial \Psi}{\partial n} = \frac{\partial \Psi}{\partial z} = 0 \quad \text{on } S_F$$




$\Psi$

basis flow

Total potential: $\Phi = \Psi + \varphi - Ux$

wave flow $<< \Psi$

free stream

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The Neumann-Kelvin linearization assumes that the perturbation potential is small compared to the free stream. This assumption is not very good, especially near the bow and stern of a ship where the perturbation velocity is equal and opposite to that of the free stream.

A better linearization for ships with forward motion is to divide the total potential into the free stream, perturbation, and basis flow potentials. The basis flow is usually taken to be the solution past the hull with the free surface treated as rigid walls. Since this problem can be solved by taking a mirror image of the hull below the waterline, this basis flow is also known as the "double-body" flow.

# Basis Flow Solution

After discretization, solve basis flow as shown for bodies in unbounded fluid.

Body Boundary Condition:

$$\frac{\partial \Psi}{\partial n} = U\, n_x$$

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The double-body basis is a special case of a problem we have already seen. A stationary non-lifting body in an unbounded free stream is simpler than all the cases that we have examined thus far.

The solution is obtained after the body is discretized and before proceeding to the wave flow. Note that the panels on the free surface are not needed for the solution of the basis flow.

# Wave Flow Body Boundary Conditions

$$\frac{\partial\varphi}{\partial n}=\frac{\partial\vec{\delta}}{\partial t}\cdot\vec{n}$$

linearized about basis flow, for small motions:

$$\frac{\partial\varphi}{\partial n}=\sum_{j=1}^{6}\left(\frac{d\xi_j}{dt}n_j+\xi_j m_j\right)$$

$$(n_1,n_2,n_3)=\vec{n} \qquad (m_1,m_2,m_3)=(\vec{n}\cdot\nabla)(\hat{i}U-\nabla\Psi)$$

$$(n_4,n_5,n_6)=\vec{x}\times\vec{n} \qquad (m_4,m_5,m_6)=(\vec{n}\cdot\nabla)[\vec{x}\times(\hat{i}U-\nabla\Psi)]$$



Since the forcing due to the free stream is accounted for in the basis flow, the body boundary condition for the wave flow component of the solution includes only the normal velocity due to the body unsteady motions.

Taking a Taylor expansion about the mean body position, and ignoring higher order terms, a linear body boundary condition is derived. As before, the combined translation/rotation vector is used to describe the body motions.

The m-terms provide a coupling between the basis flow and the unsteady wave solution, and their evaluation is important, especially near the ends of the ship.

# Wave Flow Free Surface Boundary Conditions

linearized about $\Psi$ and applied at z=0

Dynamic

$$\frac{\partial \varphi}{\partial t} - U\frac{\partial \varphi}{\partial x} + \nabla\Psi \cdot \nabla\varphi = -g\zeta + U\frac{\partial \Psi}{\partial x} - \frac{1}{2}\nabla\Psi \cdot \nabla\Psi$$

Kinematic

$$\frac{\partial \zeta}{\partial t} - U\frac{\partial \zeta}{\partial x} + \nabla\Psi \cdot \nabla\zeta = \frac{\partial^2 \Psi}{\partial z^2}\zeta + \frac{\partial \varphi}{\partial z}$$

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The free surface boundary conditions are linearized assuming that the wave flow is small compared to the basis flow. As with the Kelvin condition, these linearized boundary conditions are applied at the z=0 plane.

## Numerical Solution

**For wave flow, simultaneously solve:**

- Boundary integral equation
- Kinematic FSBC
- Dynamic FSBC
- Body boundary condition
- Equations of motion

**To obtain:**

- Potential on body and free surface
- Normal Velocity on body and free surface
- Wave elevation
- Body motions



Rankine panel methods do not have the free surface boundary conditions satisfied automatically from the choice of Green function, and hence they need to be solved simultaneously with the integral equation, equations of motion, and body boundary condition.

## Non-Linear Methods

- Higher order free surface boundary conditions
- Body-exact formulations
- Iterative linearization about a wave solution



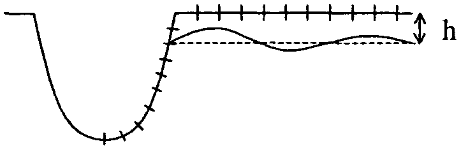
• There are cases when the linearization of the free surface conditions is not sufficient. Computation of higher order solutions is essential for some problems such as drift motions, slamming, etc. It is possible to use Rankine panel methods to solve the second order free surface boundary condition, but this would, in general, no longer involve a system of linear equations. The solution would therefore need to be found using some sort of non-linear solver.

• Body-exact methods discretize the body at its exact position at each time step, thus eliminating the error associated with the linearization of the free surface boundary conditions for large body motions. This can be very important, as seen from the inconsistencies that result when the body is only discretized below the z=0 plane.

• Taking the linearization about the double-body basis flow one step further, it is possible to obtain the linear solution and linearize the free surface conditions about that.solution. Linearizing the flow iteratively about the previous solution, the full non-linear free surface conditions should be satisfied when convergence is reached. This approach is practical only for the steady flow problem, but even for the unsteady problem several methods exist that linearize the flow about flows such as the steady wave solution or the incident wave. With these methods it is usually necessary to discretize the body and the free surface after each iteration, thus adding to the computational load. An exception is for raised panel methods, discussed later.

# Raised Panel Methods

Panels above z=0 plane




- No free surface discretization necessary at each iteration
- Influence coefficients of free surface panels to body collocation points calculated only once
- Due to distance, h, the velocity field induced in the fluid domain from each panel is smoother



One successful way of implementing the body-exact iterative linearization about a basis wave flow is by using a "raised panel" method. Such methods place singularity distributions at a distance above the z=0 plane, with the collocation points still on the free surface.

The benefits of such methods are that the free surface panels do not have to be re-created at each iteration, and the free surface to body influence coefficients need only be calculated once. The method also has nice numerical properties since the infinite velocities which are self-induced on each free surface panel are no longer in the fluid domain. In addition, the process of linearizing the flow about the previous solution is made more straightforward since the flow field at the last iteration is always defined at the next estimation of the position of the free surface.


h

## Strip Theory

Derivation of:

- Hydrodynamic Coefficients
- Exciting Force and Moment

**Assumptions:**

- Linear and harmonic motions
- Viscous effects negligible

## Potential Flow Decomposition

- Time-independent and time-dependent components

$$\Phi(x, y, z; t) = [-Ux + \phi_S(x, y, z)] + \phi_T(x, y, z)e^{i\omega t}$$

- Incident, Diffraction, Radiation components

$$\phi_T = \phi_I + \phi_D + \sum_{j=1}^{6} \zeta_j \phi_j$$

$-Ux+\phi_S$ is the steady contribution with U the forward speed of the ship, $\phi_T$ is the complex amplitude of the unsteady potential, and w is the frequency of encounter in the moving reference frame. It is understood that the real part is to be taken in expressions involving $e^{i\omega t}$

$\phi_I$ is the incident wave potential, $\phi_D$ is the diffraction potential, and $\phi_j$ is the contribution to the potential from the $j^{th}$ mode of motion (1=surge, 2=sway, 3=heave, 4=roll, 5=pitch, 6=yaw)

The decomposition of the potential into the above components is convenient for the linearization of the boundary conditions, as will be seen later.

# Linearized Boundary Conditions

- Steady Perturbation Potential
  - Body BC: (applied at hull mean position)
    $$\frac{\partial}{\partial n}\left[-Ux+\phi_S\right]=0$$
  - Free Surface BC: (applied at z=0)
    $$U^2\frac{\partial^2\phi_S}{\partial x^2}+g\frac{\partial\phi_S}{\partial z}=0$$

In order to linearize the boundary conditions it is assumed that the geometry is such that the steady perturbation potential $\phi_S$ and its derivatives are small.

By assuming that the oscillatory motions are of small amplitude, the time-dependent component of the potential, $\phi_T$, and its derivatives may also be considered small.

Under these assumptions the problem may be linearized by disregarding higher-order terms as well as cross-products on both $\phi_S$ and $\phi_T$.

The above expressions for the linear boundary conditions were derived from the exact body and free surface conditions by including only linear terms and applying Taylor expansions about the mean hull position in the body BC and about the undisturbed free surface (z=0) in the free surface BC.

# Linearized Boundary Conditions

- Incident and Diffracted Potentials
  - Body BC: (applied at hull mean position)

$$\frac{\partial \phi_I}{\partial n} + \frac{\partial \phi_D}{\partial n} = 0$$

  - Free Surface BC: (applied at $z=0$) where $\phi$ is $\phi_I$ or $\phi_D$

$$\left[\left(i\omega - U\frac{\partial}{\partial x}\right)^2 + g\frac{\partial}{\partial z}\right]\phi = 0$$

# Linearized Boundary Conditions

- Radiation Potentials
  - Body BC: (applied at hull mean position) $$\frac{\partial \phi_j^0}{\partial n} = i\omega n_j$$
  - Free Surface BC: (applied at z=0) $$\left(i\omega - U\frac{\partial}{\partial x}\right)^2 \phi_j^0 + g\frac{\partial}{\partial z}\phi_j^0 = 0$$

where: $\phi_j = \phi_j^0 \quad for \quad j = 1,2,3,4$

$$\phi_5 = \phi_5^0 + \frac{U}{i\omega}\phi_3^0$$
$$\phi_6 = \phi_6^0 - \frac{U}{i\omega}\phi_2^0$$

It can be shown that the radiation body BC is given by:

$$\frac{\partial \phi_j}{\partial n} = i\omega n_j + Um_j$$

The m-terms provide a coupling between the basis flow $\phi_B$ and the time-dependent potential.

$$(m_1, m_2, m_3) = -(\vec{n}\cdot\nabla)\frac{\nabla\phi_B}{U} \qquad (m_4, m_5, m_6) = -(\vec{n}\cdot\nabla)\left(\vec{x}\times\frac{\nabla\phi_B}{U}\right)$$

For our case, where: $\phi_B = -Ux$ we have: $(m_1, m_2, m_3) = \vec{0}$, $(m_4, m_5, m_6) = (0, n_3, -n_2)$

Let $\phi_j \equiv \phi_j^0 + \frac{U}{i\omega}\phi_j^U$ , where $\phi_j^0$ is speed independent and satisfies $\frac{\partial \phi_j^0}{\partial n} = i\omega n_j$ on the body, in addition to the Laplace equation and free-surface and infinity conditions.

It then follows that: $\phi_j^U = 0 \quad for \quad j = 1,2,3,4$

and $$\phi_5^U = \phi_3^0$$
$$\phi_6^U = -\phi_2^0$$

# Pressure Linearization

- From Bernoulli:

$$p = -\rho\left(i\omega - U\frac{\partial}{\partial x}\right)\phi_T e^{i\omega t} - \underbrace{\rho g(\zeta_3 + \zeta_4 y - \zeta_5 x)e^{i\omega t}}_{\text{buoyancy term ignored}}$$

(included in hydrostatic restoring coefficient)

Hydrodynamic force and moment:

$$H_j = -\rho\iint_S n_j\left(i\omega - U\frac{\partial}{\partial x}\right)\phi_T ds$$

(integration over the mean position of hull)

Similarly to the boundary conditions, the pressure is expanded as a Taylor series about the undisturbed position of the hull and the expression is linearized by neglecting quadratic and higher order terms in $\phi_S$ and $\phi_T$.

The hydrodynamic forces and moments, H, include the exciting forces as well as the forces due to the ship motions (added mass and damping forces). They do dot include the hydrostatic restoring forces which are included elsewhere.

# Hydrodynamic Forces

$$H_j = F_j + \sum_{k=1}^{6} T_{jk}\zeta_k$$

- Exciting Force & Moment

$$F_j = -\rho \iint_S n_j \left( i\omega - U\frac{\partial}{\partial x} \right)(\phi_I + \phi_D) ds$$

- Radiation Force & Moment

$$T_{jk} = -\rho \iint_S n_j \left( i\omega - U\frac{\partial}{\partial x} \right)\phi_k ds = \omega^2 A_{jk} - i\omega B_{jk}$$

- Need A, B, F to get equation of motion

$T_{jk}$ is the hydrodynamic force in the j$^{th}$ direction, due to a unit oscillatory displacement in the k$^{th}$ direction.

The real and imaginary parts of this force is proportional to the added mass and damping coefficients respectively. These coefficients, along with the exiting force, will be expressed in terms of integrals of the sectional (2D) coefficients over the length of the hull.

The equation of motion of the ship will then be fully specified:

$$\sum_{k=1}^{6} \left[ -\omega^2 \left( M_{jk} + A_{jk} \right) + i\omega B_{jk} + C_{jk} \right] \zeta_k = F_j$$

# Radiation Forces

- Variant of Stokes' Theorem

$$\iint_S n_j U \frac{\partial \phi}{\partial x} ds = U \int_S m_j \phi \, ds - U \int_{C_A} n_j \phi \, dl$$

- From which:

$$T_{jk} = \underbrace{-\rho i\omega \int_S n_j \phi_k ds}_{T_{jk}^0} + U\rho \iint_S m_j \phi_k ds - U\rho \int_{C_A} n_j \phi_k dl$$

- Use the decomposition of the radiation potential to express $T_{jk}$ in terms of speed-independent terms

In deriving the variant of Stokes' theorem, a small angle between the waterline and the x-axis is assumed. S is the hull surface forward of the cross section $C_A$.

As we did for the radiation potential, we can divide the hydrodynamic force into speed-independent and speed-dependent components. The speed-independent components are defined as follows:

$$T_{jk}^0 \equiv -\rho i\omega \int_S n_j \phi_k^0 ds \qquad t_{jk}^A = -\rho i\omega \int_{C_A} n_j \phi_k^0 ds$$

Then, using the properties of the radiation potential, we have:

for j,k=1,2,3,4:

$$T_{jk} = T_{jk}^0 + \frac{U}{i\omega} t_{jk}^A$$

for j=1,2,3,4:

$$T_{j5} = T_{j5}^0 + \frac{U}{i\omega} T_{j3}^0 + \frac{U}{i\omega} t_{j5}^A - \frac{U^2}{\omega^2} t_{j3}^A$$

$$T_{j6} = T_{j6}^0 - \frac{U}{i\omega} T_{j2}^0 + \frac{U}{i\omega} t_{j6}^A + \frac{U^2}{\omega^2} t_{j2}^A$$

for k=1,2,3,4:

$$T_{5k} = T_{5k}^0 - \frac{U}{i\omega} T_{3k}^0 + \frac{U}{i\omega} t_{5k}^A$$

$$T_{6k} = T_{6k}^0 + \frac{U}{i\omega} T_{2k}^0 + \frac{U}{i\omega} t_{6k}^A$$

and finally:

$$T_{55} = T_{55}^0 + \frac{U^2}{\omega^2} T_{33}^0 + \frac{U}{i\omega} t_{55}^A - \frac{U^2}{\omega^2} t_{53}^A$$

$$T_{66} = T_{66}^0 + \frac{U^2}{\omega^2} T_{22}^0 + \frac{U}{i\omega} t_{66}^A - \frac{U^2}{\omega^2} t_{62}^A$$

# Strip Theory Approximations

- Length >> Beam, Draft

$$ds = d\xi dl \Rightarrow T_{jk}^{0} = -\rho i\omega \int_{L} \int_{C_x} n_j \phi_k^0 dl d\xi = \int_{L} t_{jk} d\xi$$

$$\frac{\partial}{\partial x} << \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$$

$$n_1 << n_2, n_3 \Rightarrow \begin{cases} n_j = N_j & (j = 2,3,4) \\ n_5 = -xN_3 \\ n_6 = xN_2 \end{cases}$$

$N$: 2D normal
$n$: 3D normal

$$\omega >> U\left(\frac{\partial}{\partial x}\right)$$

The above approximations are all consistent with the trip theory assumption of a long and slender ship.

The last condition, which states that the frequency of encounter is high, requires that the maximum wave length is of the same order as the ship's beam. This enables us to simplify the radiation potential free surface condition so that $\phi^0$ is indeed speed-independent as assumed.

Under these assumptions, the 3D Laplace equation and the boundary conditions reduce to the 2D Laplace equation with the corresponding 2D boundary conditions.

# Radiation Forces in terms of 2D hydrodynamic coefficients

$$T_{22}^0 = \int t_{22} d\xi \qquad T_{33}^0 = \int t_{33} d\xi \qquad T_{44}^0 = \int t_{44} d\xi$$

$$T_{26}^0 = T_{62}^0 = \int \xi t_{22} d\xi \qquad T_{35}^0 = T_{53}^0 = -\int \xi t_{33} d\xi \qquad T_{24}^0 = T_{42}^0 = \int t_{24} d\xi$$

$$T_{66}^0 = \int \xi^2 t_{22} d\xi \qquad T_{55}^0 = \int \xi^2 t_{33} d\xi \qquad T_{46}^0 = T_{64}^0 = \int \xi t_{24} d\xi$$

All the rest $T_{jk}^0=0$, for ships with lateral symmetry

where:
$$t_{jj} = -\rho i \omega \int_{C_x} N_j \phi_j^0 dl = \omega^2 a_{jj} - i\omega b_{jj} \quad for \quad j = 2,3,4$$

$$t_{24} = -\rho i \omega \int_{C_x} N_2 \phi_4^0 dl = \omega^2 a_{24} - i\omega b_{24}$$

From the assumptions of strip theory, the two-dimensional radiation potential at each section, $\psi_k$, is equal to the three dimensional potential $\phi_k^0$ for sway, heave and roll:

$$\phi_k^0 = \psi_k \quad for \quad k = 2,3,4$$

In addition, from the hull condition, we have for pitch and yaw:

$$\phi_5^0 = -x\psi_3 \quad and \quad \phi_6^0 = x\psi_2$$

and

$$\phi_1^0 << \phi_k^0 \quad for \quad k = 2 \ldots 6$$

The above relations were used in conjunction with the expressions for the sectional radiation forces and the strip theory approximations to get the zero-speed radiation forces in terms of the 2D forces.

So from all the above, and from the relation between the speed-independent and speed-dependent components of the radiation force, we have all we need in order to express the added mass and damping coefficients in terms of the sectional two-dimensional added mass and damping.

# Incident Wave Exciting Forces

$$\left.\begin{aligned} F_j^I &= -\rho\iint_S n_j\left(i\omega - U\frac{\partial}{\partial x}\right)\phi_I ds \\ \phi_I &= \frac{ig\alpha}{\omega_0}e^{-ik(x\cos\beta - y\sin\beta)}e^{kz} \end{aligned}\right\} \Rightarrow F_j^I = -\rho i\omega_0\iint_S n_j\phi_I ds$$

(Froude-Kriloff force and moment)

$\alpha$: wave amplitude

k: wave number

$\beta$: heading angle

$\omega_0$: wave frequency, related to frequency of encounter by $\omega_0 = \omega + kU\cos\beta$

# Diffraction Forces

- Using:

$$F_j^D = -\rho \iint_S n_j \left( i\omega - U \frac{\partial}{\partial x} \right) \phi_D ds$$

  - Stokes' theorem
  - Hull BC
  - Green's 2nd identity

we get:

$$F_j^D = \rho \iint_S \left( \phi_j^0 - \frac{U}{i\omega} \phi_j^U \right) \frac{\partial \phi_I}{\partial n} ds + \frac{\rho U}{i\omega} \int_{C_A} \phi_j^0 \frac{\partial \phi_I}{\partial n} dl$$

The same form of Stokes' theorem that was used for the radiation forces earlier is applied to the diffraction forces.

The hull condition for the radiation potentials is then used to get products of potentials and normal velocities.

Green's identity, which involves such products, is then used to eliminate the radiation normal velocities (they get substituted by the diffraction normal velocities)

The hull boundary condition is then used to replace the diffraction normal velocities by the negative of the incident wave normal velocities which are known.

Use may then be made of the relations between the speed-independent and speed-dependent components of the radiation potentials to get an expression involving only the speed-independent components.

Finally, the incident wave potential, which is a known quantity, may be substituted.

# Excitation Forces in terms of 2D sectional forces

$$F_1 << F_k \quad k = 2 \ldots 6$$

$$F_j = \rho\alpha \int_L (f_j + h_j) d\xi + \rho\alpha \frac{U}{i\omega} h_j^A \quad j = 2,3,4$$

$$F_5 = -\rho\alpha \int_L \left[ \xi (f_3 + h_3) + \frac{U}{i\omega} h_3 \right] d\xi - \rho\alpha \frac{U}{i\omega} x_A h_3^A$$

$$F_6 = \rho\alpha \int_L \left[ \xi (f_2 + h_2) + \frac{U}{i\omega} h_2 \right] d\xi + \rho\alpha \frac{U}{i\omega} x_A h_2^A$$

where:

$$f_j(x) = g e^{-ikx\cos\beta} \int_{C_x} N_j e^{iky\sin\beta} e^{kz} dl \quad (2D\ Froude-Kriloff)$$

$$h_j(x) = \omega_0 e^{-ikx\cos\beta} \int_{C_x} (iN_3 - N_2 \sin\beta) \times e^{iky\sin\beta} e^{kz} \varphi_j^0 dl \quad (2D\ diffraction)$$

The excitation forces in terms of the sectional Froude-Kriloff and diffraction forces are derived from the previously derived expressions by making use of the strip theory approximations.