MIT Department of Mechanical Engineering

2.25 Advanced Fluid Mechanics

Problem 10.16

This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin

Consider the two-dimensional, steady, non-viscous flow of an incompressible fluid, with no body forces present. The flow has vorticity.

a) Show that the vorticity remains constant on each streamline.

b) Show that the stream function is governed by the equation

$$\frac{\partial \psi}{\partial y}\left(\frac{\partial^3 \psi}{\partial x^3}+\frac{\partial^3 \psi}{\partial x \partial y^2}\right)=\frac{\partial \psi}{\partial x}\left(\frac{\partial^3 \psi}{\partial y^3}+\frac{\partial^3 \psi}{\partial x^2 \partial y}\right) \tag{10.16a}$$

Solution:

a) First let us consider how many components the vorticity vector $\bar{\omega}$ has for this two dimensional flow. The vorticity vector is defined

$$\bar{\omega} = \nabla \times \bar{v} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{e}_x + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{e}_y + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{e}_z \tag{10.16b}$$

Since the z-velocity is zero, $w = 0$, and there are no gradients along the z-direction, $\frac{\partial}{\partial z} = 0$, we see that only the component of vorticity along the z-direction can be non-zero, $\bar{\omega} = \omega_z \hat{e}_z$. Hence we only need to consider how the quantity ω_z changes along a streamline to prove that the vorticity remains constant on it.

The governing equations of motion for an incompressible fluid in a two-dimensional flow are

$$\nabla \cdot \bar{v} = 0 \tag{10.16c}$$

$$\frac{D\bar{v}}{Dt} = \bar{g} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \bar{v} \tag{10.16d}$$

Or alternatively

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{10.16e}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{10.16f}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = g_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{10.16g}$$

Although we have been told that the flow is steady and free from body forces, we seek to derive as general a result as possible, so we include them in the following derivation. First let us take the curl of our two-dimensional momentum equation, Eq. (10.16d):

$$\nabla \times \left\{\frac{D\bar{v}}{Dt} = \bar{g} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \bar{v}\right\} \tag{10.16h}$$

This operation is also the same as taking the cross derivatives of Eq. (10.16f) and (10.16g) and then subtracting them. More precisely

$$\nabla \times \left\{\frac{D\bar{v}}{Dt} = \bar{g} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \bar{v}\right\} = \left(\frac{\partial}{\partial x}\hat{e}_y - \frac{\partial}{\partial y}\hat{e}_x\right) \cdot \left\{\frac{D\bar{v}}{Dt} = \bar{g} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \bar{v}\right\}\hat{e}_z \tag{10.16i}$$

Taking the first term on the right hand side of Eq. (10.16i), we have

$$\frac{\partial}{\partial x}\left\{\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=g_y-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\right\} \tag{10.16j}$$

which is equal to

$$\frac{\partial^2 v}{\partial t\partial x}+\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+u\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+v\frac{\partial^2 v}{\partial x\partial y}=\frac{\partial g_y}{\partial x}-\frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial y}+\nu\left(\frac{\partial^3 v}{\partial x^3}+\frac{\partial^3 v}{\partial x\partial y^2}\right) \tag{10.16k}$$

Taking now the first term on the right hand side of Eq. (10.16i), we have

$$\frac{\partial}{\partial y}\left\{\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=g_x-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\right\} \tag{10.16l}$$

which is equal to

$$\frac{\partial^2 u}{\partial t\partial y}+\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}+u\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}+v\frac{\partial^2 u}{\partial y^2}=\frac{\partial g_x}{\partial y}-\frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial y}+\nu\left(\frac{\partial^3 u}{\partial x^2\partial y}+\frac{\partial^3 u}{\partial y^3}\right) \tag{10.16m}$$

Combining Eq. (10.16k) and (10.16m) into Eq. (10.16i), we note that the cross-derivatives of pressure cancel and that if $\bar{g}$ is spatially uniform any gradients in g are identically zero, and we obtain

$$\frac{\partial^2 v}{\partial t\partial x}-\frac{\partial^2 u}{\partial t\partial y}+\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}+u\left(\frac{\partial^2 v}{\partial x^2}-\frac{\partial^2 u}{\partial x\partial y}\right)+v\left(\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 u}{\partial y^2}\right)=\nu\left(\frac{\partial^3 v}{\partial x^3}-\frac{\partial^3 u}{\partial x^2\partial y}+\frac{\partial^3 v}{\partial x\partial y^2}-\frac{\partial^3 u}{\partial y^3}\right) \tag{10.16n}$$

This result may be suitably rearranged to obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{\partial v}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=$$
$$\nu\left(\frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right)$$

Recalling our definitions from Eq. (10.16b) and (10.16e), we substitute these expressions into the above equation to obtain

$$\frac{\partial \omega_z}{\partial t}+u\frac{\partial \omega_z}{\partial x}+v\frac{\partial \omega_z}{\partial y}=\nu\left(\frac{\partial^2 \omega_z}{\partial x^2}+\frac{\partial^2 \omega_z}{\partial y^2}\right) \tag{10.16o}$$

or alternatively, we can write Eq. (10.16o) as

$$\frac{D\omega_z}{Dt} = \nu \nabla^2 \omega_z \tag{10.16p}$$

This result reveals, that if a flow is non-viscous, (*i.e.* $\nu = 0$), $D\omega_z/Dt = 0$ and hence the vorticity of a material element will not change as it moves along a streamline, so the vorticity remains constant on each streamline.

b) Recall that the stream function $\psi(x, y)$ is related to the velocity field by the equations

$$u = \frac{\partial \psi}{\partial y} \quad \& \quad v = -\frac{\partial \psi}{\partial x} \tag{10.16q}$$

If we substitute Eq. (10.16q) into our definition for ω_z, we have the relation

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = -\nabla^2 \psi \tag{10.16r}$$

Substituting this result into Eq. (10.16o) with $\nu = 0$ and $\partial/\partial t = 0$ since we have steady flow, we obtain

$$-\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) = 0 \tag{10.16s}$$

which can be rewritten to obtain our final result

$$\boxed{\frac{\partial \psi}{\partial y}\left(\frac{\partial^3 \psi}{\partial x^3} + \frac{\partial^3 \psi}{\partial x \partial y^2}\right) = \frac{\partial \psi}{\partial x}\left(\frac{\partial^3 \psi}{\partial y^3} + \frac{\partial^3 \psi}{\partial x^2 \partial y}\right)} \tag{10.16t}$$

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Problem Solution by TJO, Fall 2010

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