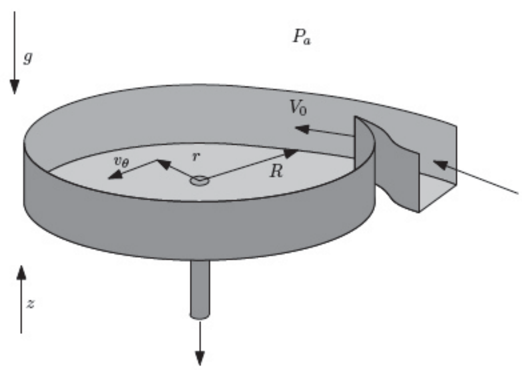
MIT Department of Mechanical Engineering
2.25 Advanced Fluid Mechanics

Problem 10.11

This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin



The steady sink flow in the sketch is set up by injecting water tangentially through a narrow channel near the periphery and letting it drain through a hole at the center. The vessel has a radius R. At the point of injection, the water has a velocity V and depth h_0; the width of the injection channel, b, is small compared with R. In what follows, we consider the region of the flow not too close to the drain, and assume that everywhere in that region (i) the flow is essentially incompressible and inviscid, (ii) the radial velocity component $|v_r|$ i small compared with the circumferential velocity component v_{theta}, and (iii) the water depth does not differ much from its value h_0 at the periphery.

- (a) Starting with Kelvin's theorem on circulation, show that

$$v_\theta = \frac{VR}{r}. \tag{10.11a}$$

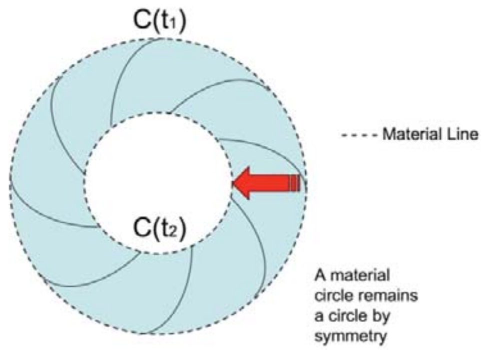
  This equation states that the angular momentum of a fluid particle remains constant in this flow. Is the angular momentum of a particle always constant? Why is it constant in this case.

- (b) Obtain result (a) from Helmoltz's vortex laws.
- (c) Obtain the result of (a) directly from Euler's equation of motion.
- (d) Show that the assumption that $|v_r| \ll v_\theta$ is satisfied if $b \ll R$.
- (e) Derive an expression for the actual distribution of water depth, given the velocity distribution of part (a), and show that the water depth is essentially constant, as we assume, provided that

$$\left(\frac{r}{R}\right)^2 \gg \frac{V^2}{2gh_0}. \tag{10.11b}$$

Solution:

- (a)



Using Kelvin's Theorem with $\rho = Const$ and $\mu = 0$, and using a circular material line as shown in the figure,

$$\Gamma_{C(t_2)} = v_\theta(r_m(t_2))r_m(t_2)2\pi = v_\theta(r_m(t_1))r_m(t_1)2\pi = \Gamma_{C(t_1)}, \tag{10.11c}$$

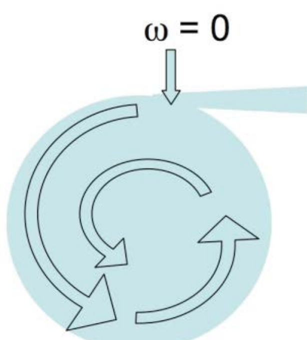
since $\frac{D\Gamma}{Dt}$. Now, let's choose $r_m(t_1) = R$ and $r_m(t_2) = r$. At these positions the velocities are U and $v_\theta(r)$ respectively . Then,

$$UR = v_\theta(r)r, \tag{10.11d}$$

then,

$$v_\theta = \frac{UR}{r}. \tag{10.11e}$$

- (b)



Since the fluid starts with null vorticity $\omega = 0$, and $\frac{D\omega}{Dt} = 0$ (Helmholtz's Vorticity Equation), then it has to remain null as the particle travels through the container, then,

$$\omega = \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} = 0, \tag{10.11f}$$

then, after integrating,

$$v_\theta = \frac{C}{r}, \tag{10.11g}$$

but $v_\theta(r=R)=U$, then $C=UR$, finally

$$v_\theta(r)=\frac{UR}{r}. \tag{10.11h}$$

- (c) Since $|v_r| \ll |v_\theta|$, we know, from Euler in cylindrical cordinates, that

$$\frac{\partial p}{\partial r}\sim\rho\frac{v_\theta^2}{r} \tag{10.11i}$$

and, since (from Helmholtz) $\omega=0$, then from Bernoulli,

$$p+\frac{1}{2}\rho v_\theta^2=Const, \tag{10.11j}$$

then, differentiating this equation, we can get the value of the pressure derivative,

$$\frac{\partial p}{\partial r}+\rho v_\theta\frac{\partial v_\theta}{\partial r}=0, \tag{10.11k}$$

then equating the value of the derivatives,

$$\rho\frac{v_\theta^2}{r}=-\rho v_\theta\frac{\partial v_\theta}{\partial r}, \tag{10.11l}$$

then, we obtain,

$$\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}=0, \tag{10.11m}$$

as before. Hence,

$$v_\theta(r)=\frac{UR}{r}. \tag{10.11n}$$

- (d) By continuity,

$$Ubh_0=-v_r 2\pi r h_0, \tag{10.11o}$$

or

$$|v_r|=\frac{Ub}{2\pi r}=\frac{UR}{r}\frac{b}{2\pi R}, \tag{10.11p}$$

where the second term $\frac{b}{2\pi R}\ll 1$ and then,

$$|v_r|\ll 1. \tag{10.11q}$$

- (e) Assuming a $2D$ flow,

$$\frac{\partial p}{\partial z}=-\rho g, \tag{10.11r}$$

where $p(z=h(r))=p_{atm}$, then

$$p = p_{atm} + \rho g(h(r) - z). \qquad (10.11s)$$

Also, from Euler-n,

$$\frac{\partial p}{\partial r} = \rho\frac{v_\theta^2}{r} = \rho\frac{U^2R^2}{r^3} = \rho g\frac{dh}{dr}, \Rightarrow \frac{dh}{dr} = \frac{U^2R^2}{gr^3}, \qquad (10.11t)$$

then after integrating, and dividing by h_0,

$$\frac{(h_0 - h(r))}{h_0} = \frac{U^2R^2}{2gh_0}\left(\frac{1}{r^2} - \frac{1}{R^2}\right), \qquad (10.11u)$$

but since $r \ll R$,

$$\frac{(h_0 - h(r))}{h_0} = \frac{U^2R^2}{2gh_0}\frac{1}{r^2}, \qquad (10.11v)$$

then $h \sim h_0$ if

$$\frac{U^2}{2gh_0}\frac{R^2}{r^2} \ll 1, \Rightarrow \left(\frac{r}{R}\right)^2 \gg \frac{U^2}{2gh_0}. \qquad (10.11w)$$

Note: We could also obtain the exact (Potential Flow) solution without assuming $|v_r| \ll |v_\theta|$ by combining a sink and an irrotational vortex,

$$\Phi = UR\theta - \frac{Ub}{2\pi}\ln r, \Rightarrow \underline{V} = \nabla\Phi = \frac{UR}{r}\underline{\hat{i}_\theta} - \frac{Ub}{2\pi r}\underline{\hat{i}_r}, \qquad (10.11x)$$

where $\nabla = \underline{\hat{i}_r}\frac{\partial}{\partial r} + \underline{\hat{i}_\theta}\frac{1}{r}\frac{\partial}{\partial\theta}$, and then the Bernoulli constant is

$$p + \frac{1}{2}\frac{U^2R^2}{r^2}\left(1 + \left(\frac{b}{2\pi R}\right)^2\right). \qquad (10.11y)$$

□

Problem Solution by ATP/MC, Fall 2008

