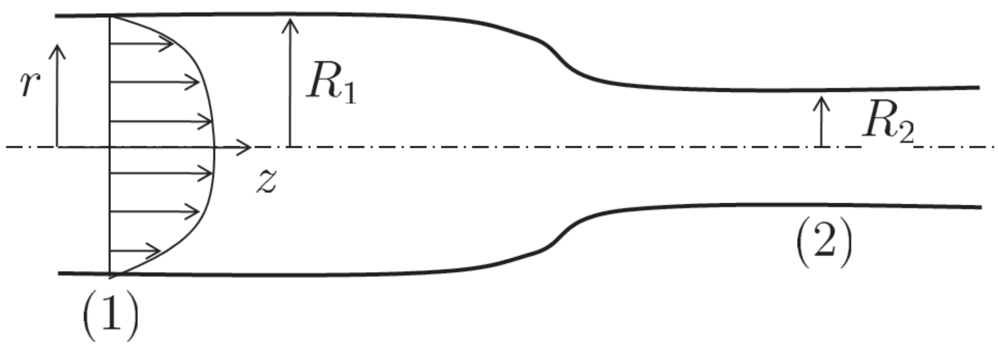
2.25 Advanced Fluid Mechanics

Problem 10.05

This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin



An inviscid, incompressible fluid flows steadily through a circular pipe with a contraction. At the entrance section, the velocity is purely in the axial direction and is given by :

$$u_1(r) = V_o \left(1 - \left(\frac{r}{R_1}\right)^2\right)$$

(a) What does the vorticity field look like at the entrance section?

(b) What is the velocity profile at the exit?

Solution:

(a) In the cylindrical coordinate, the vorticity is defined as

$$\omega = \nabla \times \mathbf{V} \tag{10.05a}$$
$$\nabla = \frac{\partial}{\partial r}\hat{\mathbf{e}_r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\mathbf{e}_\theta} + \frac{\partial}{\partial z}\hat{\mathbf{e}_z} \tag{10.05b}$$

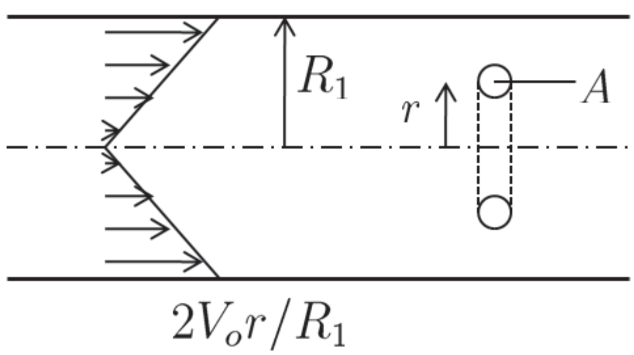
Since the flow is axially symmetric, $v_\theta = \partial/\partial\theta = 0$. And the radial velocity in this problem is zero, i.e., $v_r = 0$. Therefore,

$$\omega = -\frac{\partial v_z}{\partial r}\hat{\mathbf{e}_\theta}, \qquad where \quad v_z = u \tag{10.05c}$$

Substituting the given velocity profile gives

$$\boxed{\omega = V_o \frac{2r}{R_1^2}\hat{\mathbf{e}_\theta}} \tag{10.05d}$$

The vortex distribution looks like as following. The vortex tube looks like a ring.



(b) Kelvin's Circulation Theorem

The Kelvin's theorem represents that the circulation remains at a constant in an inviscid, barotropic flow with conservative body forces.

$$\frac{d\Gamma}{dt} = 0 \quad \Rightarrow \underline{\omega_1 A_1 = \omega_2 A_2 = const} \qquad follow\ same\ tube \tag{10.05e}$$

The mass conservation of the tube gives

$$A_1 2\pi r_1 = A_2 2\pi r_2 \Rightarrow \frac{\omega_1}{\omega_2} = \frac{A_2}{A_1} = \frac{r_1}{r_2} \tag{10.05f}$$

Hence, the Kelvin's theorem is

$$\frac{\omega_1}{r_1} = \frac{\omega_2}{r_2} \tag{10.05g}$$

Using the fact above and the $\omega_1 = \omega_\theta$ at section (1), the velocity at section (2) becomes

$$-\frac{\partial u_2}{\partial r} = \frac{2V_o r}{R_1^2} \Rightarrow u_2(r) = -\frac{V_o r^2}{R_1^2} + c \tag{10.05h}$$

Let's obtain the constant c by mass conservation between section (1) and (2).

$$\int_0^{R_1} V_o \left(1 - \left(\frac{r}{R_1}\right)^2\right) 2\pi r \, dr = \int_0^{R_2} \left(-\frac{V_o r^2}{R_1^2} + c\right) 2\pi r \, dr \tag{10.05i}$$

$$\frac{\pi V_o}{2} R_1^2 = 2\pi \left(-\frac{V_o R_2^4}{4R_1^2} + \frac{1}{2} c R_2^2\right) \tag{10.05j}$$

Then the constant c and the velocity profile at section (2) are

$$c = \frac{1}{2} V_o \frac{R_1^2}{R_2^2} + \frac{1}{2} V_o \frac{R_2^2}{R_1^2} \tag{10.05k}$$

$$\Rightarrow u_2(r) = -\frac{V_o}{R_1} r^2 + \frac{V_o}{2} \left(\frac{R_1}{R_2}\right)^2 \left\{1 + \left(\frac{R_2}{R_1}\right)^2\right\} \tag{10.05l}$$

□

Problem Solution by J.Kim, Fall 2009

