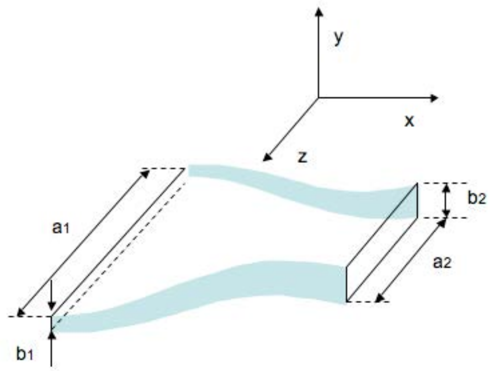
MIT Department of Mechanical Engineering
2.25 Advanced Fluid Mechanics

Problem 10.04

This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin



A steady, inviscid, incompressible flow experiences a change of cross sections between stations (1) and (2), as shown. At station (1), the velocity distribution is

$$v_x = U + ky, \qquad -\frac{b_1}{2} < y < \frac{b_1}{2}, \tag{10.04a}$$

where U is the mean flow velocity. There are no body forces acting on the fluid. Considering U, k, and the system dimensions given, determine expressions for

- (a) the vorticity at station (1),
- (b) the vorticity at station (2),
- (c) the velocity distribution at station (2),
- (d) the ration $\frac{\Delta v_x}{v_x}$ average of the total velocity excursion to the average velocity at (2), divided by the same quantity at (1).

Answer

$$\frac{\Delta v/v_{av}}{\Delta v/v_{av}} = \left(\frac{A_1}{A_1}\right)^2, \tag{10.04b}$$

where A stands for $a \cdot b$.

Solution:

- (a) The vorticity vector at station (1) is

$$\underline{\omega} = -\frac{\partial v_x}{\partial y}\hat{e}_z = -k\hat{e}_z. \tag{10.04c}$$

- (b) The x and y components of $\underline{\omega}$ are zero initially. Let's first look at how these evolve,

$$\frac{D\underline{\omega}}{Dt} = (\underline{\omega}\cdot\underline{\nabla})\underline{V}, \tag{10.04d}$$

in particular, in the x direction, (only direction not null due at the inlet and outlet)

$$\frac{D\omega_x}{Dt} = \left(\omega_x\frac{\partial}{\partial x} + \omega_y\frac{\partial}{\partial y} + \omega_z\frac{\partial}{\partial z}\right)v_x, \tag{10.04e}$$

$$\Rightarrow \frac{D\omega_x}{Dt} = \omega_z\frac{\partial v_x}{\partial z} \Rightarrow D\omega_x = \int_{station1}^{station2}\omega_z\frac{\partial v_x}{\partial z}Dt = 0. \tag{10.04f}$$

Although $\frac{Dv_x}{Dt}$ is not zero always (it is not zero specifically in the region the wall bends), we can still argue that the above integral is zero. The streamlines bend across wall-bends causing pressure differential in cross-stream direction resulting in velocity differential $\frac{\partial v_x}{\partial z}$. However, the wall bends are once concave and then convex -hence, effectively cancel each other once we integrate over the entire particle motion across the flow regime. This is a loose physical argument but we have to live with this - to escape from otherwise complicated mathematics!

$\Rightarrow \omega_x = Const = 0$

Similarly, we have the y-component:

$\omega_y = Const = 0$

Now, let's look at the evolution of ω_z,

$$\frac{D\omega_z}{Dt} = \omega_z\frac{\partial v_z}{\partial z}, \tag{10.04g}$$

Replacing $\frac{D}{Dt}$ by $\frac{d}{dt}|_m$ for derivative along a material paticle,

$$\frac{d\omega_z}{dt}|_m = \omega_z\frac{\partial v_z}{\partial z} \Rightarrow \int\frac{d\omega_z}{\omega_z}|_m = \int\frac{\partial v_z}{\partial z}dt|_m. \tag{10.04h}$$

From the figure, we can see that the variation of cross section in z-direction (i.e. variation of a) happens first (when the cross section in y direction remains the same). Similarly, the variation in y direction cross section is independent of z variation in this problem. Equation (h) only needs to be applied in the region where the variation of cross section in z-direction happens (since $\frac{\partial v_z}{\partial z}$ exists only in that region), i.e. from station 1 to say station 1'.

From station 1 to station 1' continuity gives:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0 \Rightarrow \frac{\partial v_z}{\partial z} = -\frac{\partial v_x}{\partial x}, \tag{10.04i}$$

We plug the above in (h) and do some rearrangement as below,

$$\int_1^2\frac{d\omega_z}{\omega_z}|_m = \int_1^{1'}\frac{d\omega_z}{\omega_z}|_m = -\int_1^{1'}\frac{\partial v_x}{\partial x}dt|_m = -\int_1^{1'}\frac{dt}{dx}dv_x|_m = -\int_1^{1'}\frac{dv_x}{v_x}|_m, \tag{10.04j}$$

$$\Rightarrow \frac{\omega_{z,2}}{\omega_{z,1}} = \frac{\omega_{z,1'}}{\omega_{z,1}} = \frac{v_{x,1}}{v_{x,1'}} = \frac{a_2}{a_1} \Rightarrow \omega_{z,2} = -\frac{a_2}{a_1}k. \tag{10.04k}$$

Hence, the vorticity vector at station 2 is $\underline{\omega}_2 = -\frac{a_2}{a_1}k\hat{e}_z$.

- (c) Now, at station 2,

$$\omega_z = -\frac{a_2}{a_1}k = -\frac{\partial v_x}{\partial y}, \Rightarrow v_x = \frac{a_2}{a_1}ky + C, \tag{10.04l}$$

where C is a constant of integration. Mass conservation between station 1 and 2 gives

$$a_2 \int_{\frac{b}{2}}^{\frac{b}{2}} \left(\frac{a_2}{a_1}ky + C\right) = a_1 \int_{-\frac{b_1}{2}}^{\frac{b_1}{2}} (ky + U)dy, \tag{10.04m}$$

$$\Rightarrow a_2 b_2 C = a_1 b_1 U, \Rightarrow C = \frac{a_1 b_1}{a_2 b_2} U = \frac{A_1}{A_2} U. \tag{10.04n}$$

Hence, velocity distribution at station 2 is $v_x = \frac{a_2}{a_1}ky + \frac{A_1}{A_2}U$.

- (d)First, let's calculate the requested values at 1 and 2 in order to get the ratio. First at 2, then

$$\frac{\Delta v}{v_{av}}|_2 = \frac{\left(\frac{a_2}{a_1}k\frac{b_2}{2} + \frac{A_1}{A_2}U\right) - \left(\frac{a_2}{a_1}k\left(-\frac{b_2}{2}\right) + \frac{A_1}{A_2}U\right)}{\frac{A_1}{A_2}U} = \frac{a_2 b_2 k A_2}{a_1 A_1 U} = \frac{k A_2^2}{a_1 A_1 U}. \tag{10.04o}$$

And for station 1,

$$\frac{\Delta v}{v_{av}}|_1 = \frac{\left(k\frac{b_1}{2} + U\right) - \left(-k\frac{b_1}{2} + U\right)}{U} = \frac{k b_1}{U}, \tag{10.04p}$$

then we can finally calculate the ratio,

$$\frac{\Delta v / v_{av}|_2}{\Delta v / v_{av}|_1} = \left(\frac{A_2}{A_1}\right)^2. \tag{10.04q}$$

□

Problem Solution by MK, Fall 2008

