MIT Department of Mechanical Engineering

2.25 Advanced Fluid Mechanics

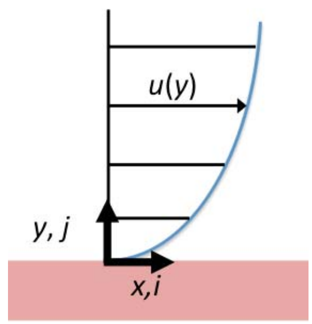
Problem 10.2

This problem is from "Advanced Fluid Mechanics Problems" by 2.25 Problem Set Solution — Problem

- (*a*) Show that if (1) and (2) are two arbitrary points in a steady, inviscid, incompressible flow in a uniform gravitational field,

$$\left(P_2 + \frac{v_2^2}{2} + \rho g y_2\right) = \left(P_1 + \frac{v_1^2}{2} + \rho g y_1\right) + \rho \int_1^2 (\underline{v} \times \underline{\omega}) \cdot d\underline{l} \tag{10.2-1}$$

  Here, y is measured up against the gravitational field, $\omega = \nabla \times \underline{v}$ is the vorticity vector and the last term represents a line integral along any path between (1) and (2) through the flow.



- (*b*) Show that if the flow in (a) is a parallel, horizontal flow, that is,

$$\underline{v} = u(y)\underline{i}, \tag{10.2-2}$$

  as shown in the sketch, it follows from the equation in (a) that the pressure distribution is the hydrostatic one,

$$P_2 + \rho g y_2 = P_1 + \rho g y_1 \tag{10.2-3}$$

- (*c*) Obtain the conclusion of (b) by using an argument based on Euler's equation in streamline form, rather than starting with the equation in part (a)

Solution:

(a) From Cauchy momentum equation, we can derive the following equations for a steady, inviscid, and incompressible flow in a uniform gravitational field.

$$\rho\frac{D\underline{v}}{Dt} = \rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot(\nabla\underline{v}) = -\nabla P + \rho(-\nabla gz) \tag{10.2-4}$$

$$\frac{\partial \not{v}}{\not{\partial} t} + \nabla\left(\frac{1}{2}\mid\underline{v}\mid^2\right) - \underline{v}\times\underline{w} = -\frac{1}{\rho}\nabla P - \nabla gz \tag{10.2-5}$$

$$\frac{1}{\rho}\nabla P + \nabla\left(\frac{1}{2}\mid\underline{v}\mid^2\right) + \nabla gz = \underline{v}\times\underline{w} \tag{10.2-6}$$

$$\int_1^2\left(\frac{1}{\rho}\nabla P + \nabla\left(\frac{1}{2}\mid\underline{v}\mid^2\right) + \nabla gz\right)\cdot dl = \int_1^2\left(\underline{v}\times\underline{w}\right)\cdot dl \tag{10.2-7}$$

Therefore, the final form is the same as

$$\left(P_2 + \frac{v_2^2}{2} + \rho g y_2\right) = \left(P_1 + \frac{v_1^2}{2} + \rho g y_1\right) + \rho\int_1^2(\underline{v}\times\underline{\omega})\cdot d\underline{l} \tag{10.2-8}$$

(b)

$$\omega = \nabla\times\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x(y) & 0 & 0 \end{vmatrix} = -\frac{\partial u_x}{\partial y}\underline{k} \tag{10.2-9}$$

$$\underline{v}\times\underline{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_x & 0 & 0 \\ 0 & 0 & -\frac{\partial u_x}{\partial y} \end{vmatrix} = u_x\frac{\partial u_x}{\partial y}\underline{j} = \nabla\left(\frac{1}{2}\mid\underline{v}\mid^2\right) \tag{10.2-10}$$

Therefore we can cancel $\nabla\left(\frac{1}{2}\mid\underline{v}\mid^2\right)$ in the LHS of eq.(10.2-6) and it results in,

$$P_2 + \rho g y_2 = P_1 + \rho g y_1 \tag{10.2-11}$$

(c)

Euler's equation in the normal direction ($\underline{e}_n$) is

$$-\rho\frac{V_s^2}{R} = -\frac{\partial P}{\partial n} + \rho g_n \tag{10.2-12}$$

Here, $R \to \infty$ because of parallel flow.

Therefore, integration from point 1 to point 2 gives again,

$$P_2 + \rho g y_2 = P_1 + \rho g y_1 \tag{10.2-13}$$

□

Problem Solution by KP and BK, Fall 2012

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