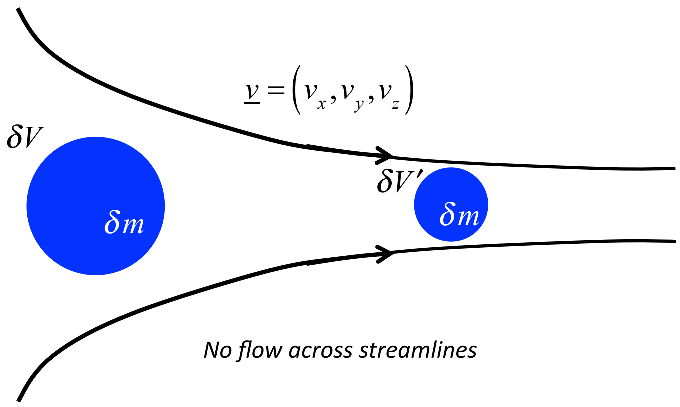
MIT Department of Mechanical Engineering

# 2.25 Advanced Fluid Mechanics

## The Continuity Equation: Conservation of Mass for a Fluid Element




Consider a fluid element with constant mass $\delta m$ and volume $\delta V$ moving in a velocity field as shown above. The streamlines are converging and the fluid element may be advected to a new position which has a higher speed. If we assume the most general case, in which the fluid element is compressible, then $\delta m$ is fixed but $\delta V$ changes. Note that:

$$\rho = \delta m/\delta V \tag{1}$$

Hence:

$$\delta V = \frac{\delta m}{\rho} = \delta m \rho^{-1} \Rightarrow \frac{D(\delta V)}{Dt} = -\frac{\delta m}{\rho^2}\frac{D\rho}{Dt} \tag{2}$$

$\Rightarrow$

$$\frac{1}{\delta V}\frac{D(\delta V)}{Dt} = -\frac{1}{\rho}\left(\frac{D\rho}{Dt}\right) \tag{3}$$

We already know that the left hand side of (3) is $\underline{\nabla} \cdot \underline{v}$, thus:

$$\frac{1}{\rho}\left(\frac{D\rho}{Dt}\right) = -\underline{\nabla} \cdot \underline{v} \tag{4}$$

Alternative way to reach the same thing is:

$$\frac{D(\delta m)}{Dt} = 0 \Rightarrow \frac{D(\rho \delta V)}{Dt} = 0 \Rightarrow \delta V \frac{D(\rho)}{Dt} + \rho \frac{D(\delta V)}{Dt} = 0 \tag{5}$$

Dividing by $\rho \delta V$ leads to:

$$\frac{1}{\rho}\frac{D(\rho)}{Dt} + \frac{1}{\delta V}\frac{D(\delta V)}{Dt} = 0 \tag{6}$$

Again knowing that volumetric rate of strain, the second term, is equal to $\underline{\nabla} \cdot \underline{v}$ gives:

$$\frac{1}{\rho}\frac{D(\rho)}{Dt} = -\underline{\nabla} \cdot \underline{v} \tag{7}$$

which is the same concluded in (4). The derived equation is mass conservation for any flow (compressible or incompressible). In the case of incompressible flows (or almost "incompressible"-Mach numbers lower than 0.3), from "incompressibility" we will have:

$$\frac{1}{\rho}\frac{D\left(\rho\right)}{Dt} \simeq 0 \tag{8}$$

which by (7) means that in incompressible flows ($Ma < 0.3$):

$$\underline{\nabla} \cdot \underline{v} \simeq 0 \tag{9}$$