Solid Body Rotation-Extra Notes

Whenever we have a coordinate rotation the following holds:
Imagine a vector $\underline{v}$:

$$\underline{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

in $x - y$ coordinate system and want to calculate the components of $\underline{v}$ in a new coordinate system $x' - y'$ which comes from θ counterclockwise rotation of $x - y$ coordinate system (Figure 1).

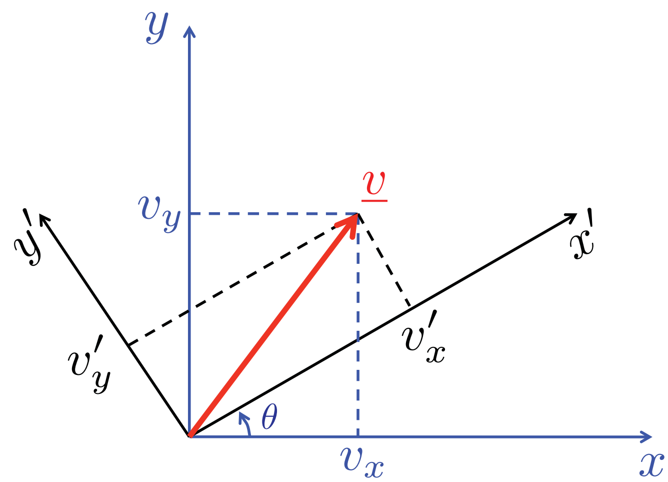

Figure 1: Coordinate system $x' - y'$ is a θ counterclockwise rotation of $x - y$.

$$\begin{bmatrix} v_{x'} \\ v_{y'} \end{bmatrix} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} \times \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

If you show the vectors in a row form rather than the column form then the matrix from of this conversion will look different:

$$\begin{bmatrix} v_{x'} & v_{y'} \end{bmatrix} = \begin{bmatrix} v_x & v_y \end{bmatrix} \times \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

It is easy to see that both of these matrix representations express the same set of equations for conversion between the two coordinate systems:

$$\begin{aligned} v_{x'} &= v_x cos\theta + v_y sin\theta \\ v_{y'} &= v_x(-sin\theta) + v_y cos\theta \end{aligned} \tag{1}$$

and if you like to get the components in the $x - y$ coordinates from the $x' - y'$ coordinate then the following holds:

$$\begin{aligned} v_x &= v_{x'} cos\theta - v_{y'} sin\theta \\ v_y &= v_{x'} sin\theta + v_{y'} cos\theta \end{aligned} \tag{2}$$

In the solid body rotation problem we have to convert components from $x - y$ coordinate to $r - \theta$ coordinates:

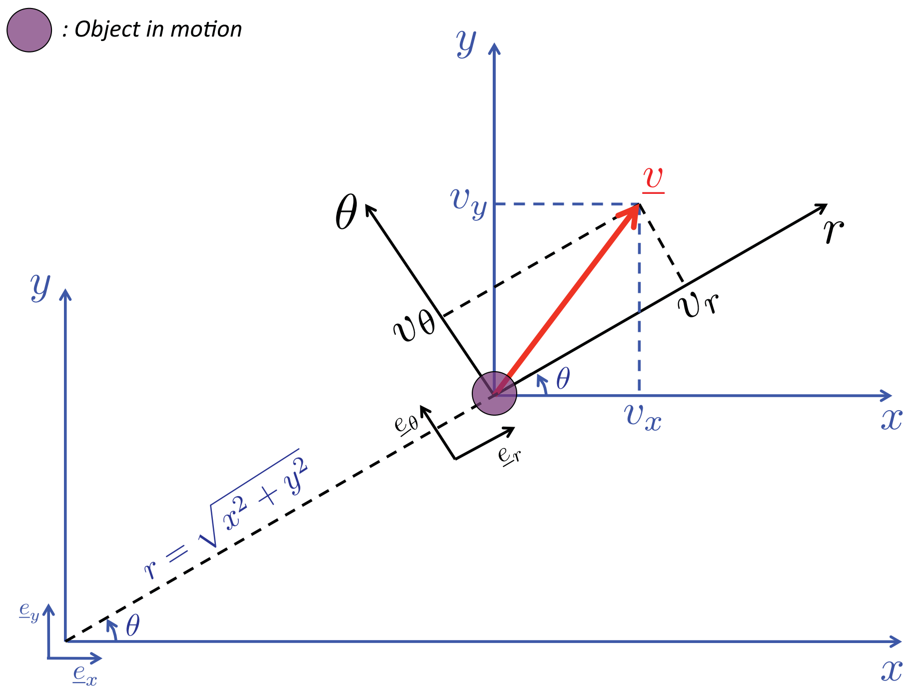

Figure 2: Coordinate system $r - \theta$ is a θ counterclockwise rotation of $x - y$. Note that here the $\underline{v}$ vector is drawn for an arbitrary case and has both r and θ components whereas in the rotation problem the r component is zero.

The $\underline{v}$ velocity vector in the polar coordinate for this problem is:

$$\underline{v} = \begin{bmatrix} v_r \underline{e}_r \\ v_\theta \underline{e}_\theta \end{bmatrix} = \begin{bmatrix} 0 \\ r\Omega \underline{e}_\theta \end{bmatrix}$$

Using equation (2) we can convert the velocity components from $r - \theta$ coordiantes back to the $x - y$ coordinates:

$$v_x = v_r cos\theta - v_\theta sin\theta = 0cos\theta - r\Omega sin\theta$$
$$v_y = v_r sin\theta + v_\theta cos\theta = 0sin\theta + r\Omega cos\theta$$

Using the fact that $y = rsin\theta$ and $x = rcos\theta$ we can simplify it to:

$$v_x = -y\Omega$$
$$v_y = x\Omega$$

Now as shown in the class after using the material derivative we can find the acceleration vector ($\underline{a} = D\underline{v}/Dt$) in the $x - y$ coordinate system:

$$\underline{a} = \begin{bmatrix} a_x \, \underline{e}_x \\ a_y \, \underline{e}_y \end{bmatrix} = \begin{bmatrix} -\Omega^2 x \, \underline{e}_x \\ -\Omega^2 y \, \underline{e}_y \end{bmatrix}$$

Now using equation (1) we can convert this into the $r - \theta$ coordinates:

$$a_r = a_x cos\theta + a_y sin\theta = -\Omega^2 x cos\theta - \Omega^2 y sin\theta$$
$$a_\theta = a_x(-sin\theta) + a_y cos\theta = \Omega^2 x sin\theta - \Omega^2 y cos\theta$$

Using the fact that $y = rsin\theta$, $x = rcos\theta$ and $cos^2\theta + sin^2\theta = 1$ we can simplify it to:

$$a_r = -r\Omega^2$$
$$a_\theta = 0$$

thus:

$$\boxed{\underline{a} = \begin{bmatrix} a_r \underline{e}_r \\ a_\theta \underline{e}_\theta \end{bmatrix} = \begin{bmatrix} -r\Omega^2 \ \underline{e}_r \\ 0 \ \underline{e}_\theta \end{bmatrix}}$$