MIT Department of Mechanical Engineering

2.25 Advanced Fluid Mechanics

Problem 1.00

This problem is from "2.25 Advanced Fluid Mechanics" by Ain Sonin

Rate of change of properties measured by a probe moving through the earth's atmosphere — plus some things about the earth and its atmosphere.

The pressure distribution in a static, constant-temperature planetary atmosphere modeled as an ideal gas is given by

$$p = p_0 e^{-z/H} \tag{1.00a}$$

where z is the altitude above a reference altitude $z = 0$, p_0 is the absolute pressure at $z = 0$, and

$$H = \frac{RT}{Mg} \tag{1.00b}$$

is a length scale that characterizes the atmosphere. Its value is determined by the strength of the gravitational acceleration and the parameters that appear in the ideal-gas equation of state,

$$p = \rho \frac{RT}{M} \; ; \tag{1.00c}$$

$R = 8.32$ JK^{-1} mol^{-1} is the universal gas constant, T is the absolute temperature (taken as constant in this model of the atmosphere), M is the molar mass of the gas (0.029 kg/mol if the gas is air), and g is the acceleration of gravity at or near the surface of the planet. For the "Standard" isothermal model of the earth's atmosphere, $T = 288$ K, $p_0 = 1.02 \times 10^5$ N/m^2 if $z = 0$ at sea level, and consequently $H = 8.43$ km. Note that the distribution given above is based on the assumption that $H \ll a$, where a is the planet's radius.

Suppose a sounding rocket or balloon equipped with a static-pressure sensor is traveling through the atmosphere with given velocity (v_x, v_y, v_z).

1. In terms of the given quantities and z, derive an expression for the rate of change of pressure recorded by the rocket's sensor.
2. Evaluate this time of change at an altitude $z = 20{,}000$ m for a rocket traveling upward through the earth's atmosphere with a direction of 30° from the vertical and a speed of 465 m/s. (Answer: -0.273 bar/min.)
3. Suppose a rocket carries instruments that measure both the instantaneous atmospheric pressure p and the rate of change of that pressure, dp/dt. Given the value of these two quantities at a particular time and p_0 and H, derive expressions for the rocket's instantaneous altitude and vertical (upward) velocity.

Additional things to think about, if you are so inclined:

4. Suppose the Earth's atmosphere is isothermal and radially symmetric around a perfectly spherical earth with radius $a = 6400$ km. What is the total mass of the Earth's atmosphere? What fraction is this of the solid and liquid parts of the planet's mass? (Answers: 5.35×10^{18} kg and 8.96×10^{-7}.)

5. Show that 99% of the Earth's atmosphere's mass resides below an altitude of 39 km.

6. If the atmosphere heats up by 10°C, by how much will the absolute pressure at sea level change? (Answer: it will not change at all.)

Solution:

1. Take material derivative of (1.00a).

$$\frac{Dp}{Dt} = \cancelto{=0}{\frac{\partial p}{\partial t}} + \mathbf{v}\cdot\nabla p = \frac{\partial p}{\partial z}\cdot\frac{\partial z}{\partial t} = -\frac{p_0}{H}\,e^{-z/H}\cdot\frac{\partial z}{\partial t}$$

where $\frac{\partial z}{\partial t} = v_z$.

$$\boxed{\frac{Dp}{Dt} = -\frac{v_z p_0}{H}\,e^{-z/H}}\;. \qquad (1.00d)$$

2. Calculate the z component of the given velocity, 465 m/s, $v_z = 465\cos 30°$. Plug in the values for v_z, z, H, and p_0 into Eq. (1.00d):

$$\boxed{\frac{Dp}{Dt}(z = 20{,}000\text{ m}) \approx -454.36\text{ N/m}^2\text{s} \approx -0.273\text{ bar/min.}}$$

(Hint: 1 bar $\equiv 10^5$ N/m^2.)

3. Given: $p_0,\ H,\ p,\ \frac{Dp}{Dt}$

Unknown: $z,\ v_z$

Equations: (1) and (1.a) ,

obtain an expression for z in terms of H, p_0, and p by rewriting (1.00a):

$$\ln\left(\frac{p}{p_0}\right) = -\frac{z}{H}$$

$$\therefore \qquad \boxed{z = H\ln\left(\frac{p_0}{p}\right)}\;.$$

Take Eq. (1.00d) and solve for v_z:

$$\frac{Dp}{Dt} = -\frac{v_z}{H}\cdot\underbrace{p_0\,e^{-z/H}}_{=p}$$

$$\frac{Dp}{Dt} = -\frac{v_z p}{H}$$

$$\therefore \qquad \boxed{v_z = -\frac{H}{p}\cdot\frac{Dp}{Dt}}\;.$$

4. Rewrite (1.00c) and combine it with (1.00a) to solve for ρ:

$$\rho = \frac{M}{RT}\,p_0\,e^{-z/H}\;. \qquad (1.00e)$$

$$\text{Mass}_{\text{atm}} = \int_V \rho\,dV = \frac{Mp_0}{RT}\int_V e^{-z/H}\,dV$$

where $dV = 4\pi r^2 dz = 4\pi(a+z)^2\,dz$.

Perform the integration by parts twice to solve the following integral:

$$\begin{aligned} \text{Mass}_{\text{atm}} &= \frac{4\pi M p_0}{RT} \int_0^\infty e^{-z/H}(z+a)^2\,dz \\ &= \frac{4\pi M p_0}{RT}(Ha^2 + 2H^2 a + 2H^3) \\ &\approx \boxed{5.35 \times 10^{18}\ \text{kg.}} \end{aligned} \tag{1.00f}$$

(Hint on Integration by parts $\left(\int u dv = uv - \int v du\right)$ When integrating by parts the first time, let $u = (z+a)^2$ and $dv = e^{-z/H}dz$. For the second time, let $u = (z+a)$ and $dv = e^{-z/H}dz$.)

The mass of the earth is known as 6×10^{24} kg.

$$\therefore \quad \boxed{\frac{\text{Mass}_{\text{atm}}}{\text{Mass}_{\text{earth}}} \approx 8.92 \times 10^{-7}.}$$

5. Perform the same integration as in (d) but with [0 39,000] as limits.

6. The absolute pressure at sea level is a measure of the total weight of the atmosphere that is sitting on top. Because the total atmospheric mass does not change, the pressure remains constant as well. Mathematically, if one integrates $\frac{dp}{dz} = \rho g$ on both sides from 0 to ∞ in terms of dz, RHS becomes $\int_0^\infty \rho dz$, which is constant. Therefore, LHS, $p(\infty) - p(0)$, should also be constant. Since $p(\infty)$ is zero by definition, $p(0)$, or the absolute pressure at sea level, becomes constant, independent of temperature change.

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