MIT Department of Mechanical Engineering

2.25 Advanced Fluid Mechanics

Kundu & Cohen 6.8

This problem is from "Fluid Mechanics" by P. K. Kundu and I. M. Cohen

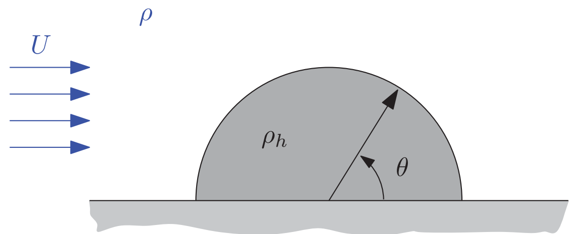
A solid hemisphere of radius a is lying on a flat plate. A uniform stream U is flowing over it. Assuming irrotational flow, show that the density of the material must be

$$\rho_h \geq \rho\left(1+\frac{33}{64}\frac{U^2}{ag}\right)$$

to keep it on the plate.

Solution:

Note that we are looking at the flow around a solid hemisphere not a semi-circle.



Due to high speed flow at the top of the sphere, we expect a low pressure at the top of the sphere. This pressure results in a lift force on the hemsiphere.

Given the velocity field, the pressure distribution at the surface of the sphere can be found using Bernoulli:

$$p(\theta) - p_a = \frac{1}{2}\rho(U^2 - v(r,\theta)^2)$$

We can then integrate the pressure at the surface of the hemisphere to find the lift force. The flow around this hemisphere is the same as that for a sphere because of symmetry about the plate. Thus, streamlines for this flow can be solved by combining the streamlines for a uniform flow and a doublet.

from Kundu & Cohen pp.192

$$\psi_{\text{hemisphere}} = \psi_{\text{sphere}} = \psi_{\text{uniform}} + \psi_{\text{doublet}} = \frac{1}{2}Ur^2\sin^2\theta - \frac{m}{r}\sin^2\theta$$
$$v_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}$$
$$v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

where m is the strength of the doublet. First, let us evaluate v_r

$$\begin{aligned} v_r &= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\left(\frac{1}{2}Ur^2 - \frac{m}{r}\right)\sin^2\theta\right] \\ &= \frac{1}{r^2\sin\theta}\left(\frac{1}{2}Ur^2 - \frac{m}{r}\right)2\sin\theta\cos\theta = \left(U - 2\frac{m}{r^3}\right)\cos\theta \qquad (1) \end{aligned}$$

Similarly for v_θ:

$$\begin{aligned} v_\theta &= -\frac{1}{r\sin\theta}\frac{\partial}{\partial r}\left[\left(\frac{1}{2}Ur^2 - \frac{m}{r}\right)\sin^2\theta\right] \\ &= -\frac{1}{r\sin\theta}\left(Ur + \frac{m}{r^2}\right)\sin^2\theta = -\left(U + \frac{m}{r^3}\right)\sin\theta \qquad (2) \end{aligned}$$

Now we must solve for the doublet strength m. We know there is a stagnation point at $r = a$ and $\theta = \pi$ (and also for $\theta = 0$) such that our velocities are zero:

$$v_r\big|_{r=a,\theta=\pi} = 0 = \left(U - 2\frac{m}{a^3}\right)$$
$$\Rightarrow m = \frac{1}{2}Ua^3$$

Now substitute this into Eqs (1) and (2)

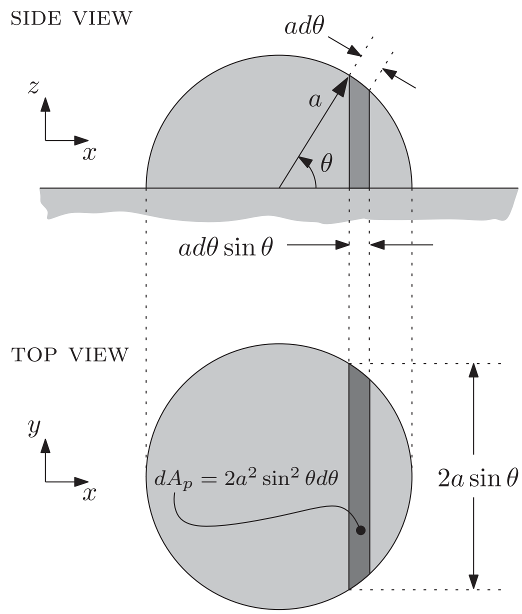
$$\boxed{\begin{aligned} v_r &= U\left[1 - \left(\frac{a}{r}\right)^3\right]\cos\theta \\ v_\theta &= -U\left[1 + \frac{1}{2}\left(\frac{a}{r}\right)^3\right]\sin\theta \end{aligned}} \qquad (3)$$

At the surface of the hemisphere $r = a$, such that $v_r = 0$ (no flux through the sphere). Thus

$$v(r,\theta)\big|_{r=a} = v_\theta(a,\theta) = -\frac{3}{2}U\sin\theta$$

Since the pressure is only a function of θ, we can solve for the lift force by integrating the pressure over the area of the hemisphere projected on the x-y plane, A_p:

$$\begin{aligned}
F_p &= \int_{A_p} p - p_a \, dA_p \\
&= \int_{A_p} \frac{1}{2}\rho\left[U^2 - \left(-\frac{3}{2}U\sin\theta\right)^2\right] dA_p \\
&= \frac{1}{2}\rho U^2 \int_0^\pi (2a^2\sin^2\theta)d\theta \\
&\quad - \frac{9}{8}\rho U^2 \int_0^\pi \sin^2\theta(2a^2\sin^2\theta)d\theta
\end{aligned}$$



From table of integrals:

$$\int \sin^2 x\,dx = \frac{1}{2}(x - \sin x\cos x) + C$$

$$\int \sin^4 x\,dx = -\frac{\sin^3 x\cos x}{4} + \frac{3}{8}(x - \sin x\cos x) + C$$

Therefore,

$$\begin{aligned}
F_p &= 2\frac{1}{2}\rho U^2 a^2\left[\frac{1}{2}(\theta - \sin\theta\cos\theta)\right]_0^\pi - \frac{9}{4}\rho a^2 U^2\left[-\frac{\sin^3\theta\cos\theta}{4} + \frac{3}{8}(\theta - \sin\theta\cos\theta)\right]_0^\pi \\
&= \frac{1}{2}\rho a^2\pi U^2 - \frac{27}{32}\pi\rho a^2 U^2 = -\frac{11}{32}\pi\rho a^2 U^2
\end{aligned}$$

The sign of this force tells us the pressure has a lifting effect (a positive pressure on an upward facing surface pushes downward). Thus $F_L = -F_p = \frac{11}{32}\pi\rho a^2 U^2$. The weight of the hemisphere is given by

$$W = \rho_h g V = \rho_h g\frac{2}{3}\pi a^3$$

which acts downward. There is also a buoyancy force given by

$$F_B = \rho g V = \rho g\frac{2}{3}\pi a^3$$

which acts upward. To keep the hemisphere on the plate we need the downward acting force W to be greater than or equal to the upward acting forces, $F_p + F_B$:

$$W \geq F_p + F_B$$

$$\rho_h g\frac{2}{3}\pi a^3 \geq \rho g\frac{2}{3}\pi a^3 + \frac{11}{32}\pi\rho a^2 U^2$$

$$\boxed{\rho_h \geq \rho\left(1 + \frac{33}{64}\frac{U^2}{ag}\right)}$$

□

Problem Solution by Tony Yu, Fall 2006

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