MIT Department of Mechanical Engineering

2.25 Advanced Fluid Mechanics

Kundu & Cohen 6.4

This problem is from "Fluid Mechanics" by P. K. Kundu and I. M. Cohen

(a) Take a plane source of strength m at point $(-a, 0)$, a plane sink of equal strength at $(a, 0)$, and superpose a uniform stream U directed along the x-axis.

(b) Show that there are two stagnation points located on the x-axis at points

$$\pm a \left(\frac{m}{\pi a U} + 1 \right)^{1/2}.$$

(c) Show that the streamline passing through the stagnation points is given by $\psi = 0$. Verify that the line $\psi = 0$ represents a closed oval-shaped body, whose maximum width h is given by the solution of the equation

$$h = a \cot \left(\frac{\pi U h}{m} \right).$$

The body generated by the superposition of a uniform stream and a source-sink pair is called a *Rankine body*. It becomes a circular cylinder as the source–sink pair approach each other.

Solution:

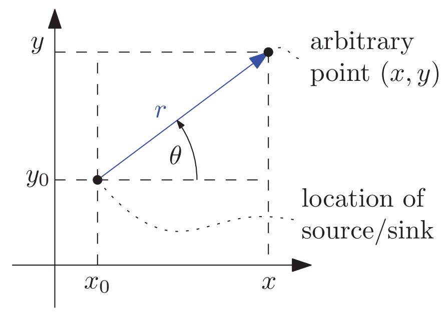
(a)

$$W(z) = W_{\text{uniform flow}} + W_{\text{source}} + W_{\text{sink}}$$

where $W = \phi + i\psi$, ϕ is the potential function, and ψ the stream function.

> Recap from Lecture: W satisfies the Laplace equation which is linear. Therefore, one can superimpose its solutions as above.

$$W_{\text{uniform flow}} = U_\infty(x+iy)$$
$$W_{\text{source}} = \frac{m}{2\pi}\ln\left(re^{i\theta}\right) = \left(\frac{m}{2\pi}\ln r + i\frac{m\theta}{2\pi}\right)$$
$$W_{\text{sink}} = -\frac{m}{2\pi}\ln\left(re^{i\theta}\right) = -\left(\frac{m}{2\pi}\ln r + i\frac{m\theta}{2\pi}\right)$$



Substitute expressions for r and θ in terms of x and y (see figure):

$$r = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$
$$\theta = \arctan\left(\frac{y-y_0}{x-x_0}\right)$$

$$W_{\text{source}} = \frac{m}{2\pi}\ln\left(re^{i\theta}\right) = \left(\frac{m}{2\pi}\ln r + i\frac{m\theta}{2\pi}\right)$$
$$W_{\text{sink}} = -\frac{m}{2\pi}\ln\left(re^{i\theta}\right) = -\left(\frac{m}{2\pi}\ln r + i\frac{m\theta}{2\pi}\right)$$

$$\Rightarrow W_{\text{total}} = \underbrace{U_\infty x + \frac{m}{4\pi}\ln\frac{(x+a)^2+y^2}{(x-a)^2+y^2}}_{\phi} + i\underbrace{\left[U_\infty y + \frac{m}{2\pi}\arctan\left(\frac{y}{x+a}\right) - \frac{m}{2\pi}\arctan\left(\frac{y}{x-a}\right)\right]}_{\psi}$$

(b) Obtain the velocity field (v_x, v_y) by invoking $\mathbf{v} = \nabla\phi$

$$v_x = \frac{\partial\phi}{\partial x} = U_\infty + \frac{m}{4\pi}\cdot\frac{\cancel{(x-a)^2+y^2}}{(x+a)^2+y^2}\left\{\frac{2(x+a)}{\cancel{(x-a)^2+y^2}} - \frac{(x+a)^2+y^2}{[(x-a)^2+y^2]^{\cancel{2}}}\cdot 2(x-a)\right\}$$

$$\Rightarrow \boxed{v_x = U_\infty + \frac{m}{2\pi}\left[\frac{x+a}{(x+a)^2+y^2} - \frac{x-a}{(x-a)^2+y^2}\right]}$$

$$v_y = \frac{\partial\phi}{\partial y} = \frac{m}{4\pi}\cdot\frac{\cancel{(x-a)^2+y^2}}{(x+a)^2+y^2}\left\{\frac{2y}{\cancel{(x-a)^2+y^2}} - \frac{(x+a)^2+y^2}{[(x-a)^2+y^2]^{\cancel{2}}}\cdot 2y\right\}$$

$$\Rightarrow \boxed{v_y = \frac{my}{2\pi}\left[\frac{1}{(x+a)^2+y^2} - \frac{1}{(x-a)^2+y^2}\right]}$$

> Alternatively, one can find $\mathbf{v}$ by using: $v_x = \frac{\partial\psi}{\partial y}$, $v_y = -\frac{\partial\psi}{\partial x}$.

Find the stagnation point(s) by finding (x, y) such that $v_x = v_y = 0$.

$$v_y = 0 \text{ at } y = 0,\ \forall x$$

Plug in $y = 0$ into v_x and find x that lets $v_x = 0$:

$$v_x(x, y=0) = U_\infty + \frac{m}{2\pi}\left(\frac{x+a}{(x+a)^2} - \frac{x-a}{(x-a)^2}\right) = 0$$

$$U_\infty + \frac{m}{2\pi}\left(\frac{1}{x+a} - \frac{1}{x-a}\right) = 0$$

$$\text{OR (after some algebra...)}: \quad x^2 - a^2 - \frac{ma}{\pi U_\infty} = 0$$

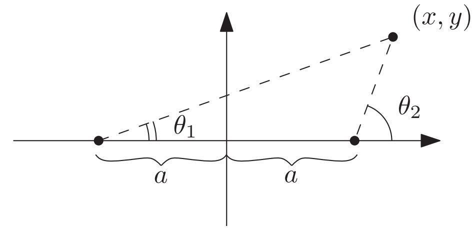
Using the quadratic formula[1],

$$x = \pm a\sqrt{1 + \frac{m}{a\pi U_\infty}}$$

(c) Going back to ψ:

$$\psi = U_\infty y + \frac{m}{2\pi}\overbrace{\arctan\left(\frac{y}{x+a}\right)}^{\theta_1} - \frac{m}{2\pi}\overbrace{\arctan\left(\frac{y}{x-a}\right)}^{\theta_2}$$

$$= U_\infty y - \frac{m}{2\pi}\underbrace{\arctan\left(\frac{2ay}{x^2+y^2-a^2}\right)}_{-(\theta_1-\theta_2)}$$



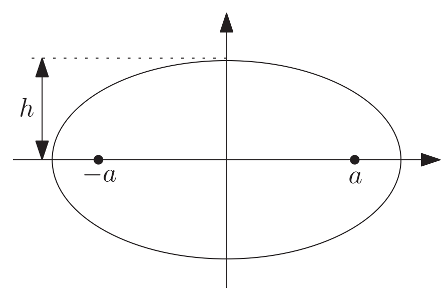
A "Rankine oval" is defined by a curve $\psi = 0$, or

$$\boxed{U_\infty y - \frac{m}{2\pi}\arctan\left(\frac{2ay}{x^2+y^2-a^2}\right) = 0} \qquad (1)$$

Maximum <u>half-width</u>, h, is obtained when $x = 0$:

$$U_\infty h = \frac{m}{2\pi}\arctan\left(\frac{2ah}{h^2-a^2}\right)$$

$$\frac{2\pi U_\infty h}{m} = \operatorname{arccot}\left(\frac{h^2-a^2}{2ah}\right)$$



$$\Rightarrow \boxed{h\left[1 - \left(\frac{a}{h}\right)^2\right] = 2a\cot\left(\frac{2\pi U_\infty h}{m}\right)} \qquad (2)$$

□

Problem Solution by Sungyon Lee, Fall 2005

[1] $ax^2 + bx + c = 0$, $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

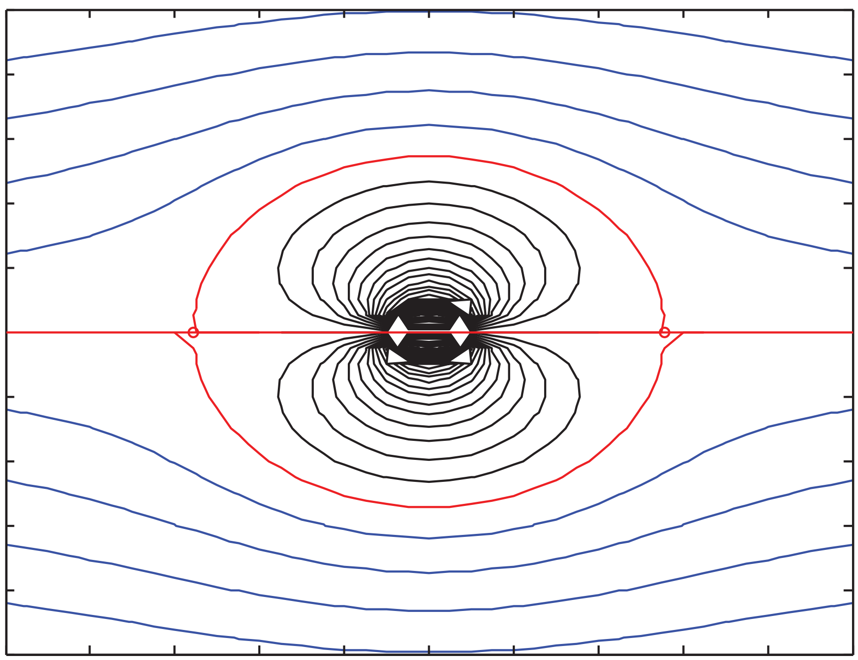


Figure 1: MATLAB® plot of streamlines for a Rankine oval.

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