

MIT Department of Mechanical Engineering
2.25 Advanced Fluid Mechanics

Kundu & Cohen 6.14

This problem is from "Fluid Mechanics" by P. K. Kundu and I. M. Cohen

In a two-dimensional constant density potential flow, a source of strength m is located a meters above an infinite plane. Find the velocity on the plane, the pressure on the plane, and the reaction force on the plane.

Solution:

Here we use the method of images and note that in the complex plane the two point sources are located at $z = \pm ia$, where $i^2 = -1$. Now the complex potential for this flow is given by

$$w = \frac{m}{2\pi} \ln(z - ia) + \frac{m}{2\pi} \ln(z + ia) - \frac{m}{2\pi} \ln a^2 \quad (1)$$

where we have included the term $\ln a^2$ to make the arguments of the logarithms dimensionless (recall that $\ln z - \ln a = \ln \frac{z}{a}$). The inclusion of this term, however, does not affect the velocity field, since it is eliminated when we take the gradient of w . The above result in Eq.(1) is similar, but not identical to Eq. (6.50) in Kundu. Why? The equation expands to

$$w = \frac{m}{2\pi} \ln(x^2 - y^2 + a^2 + i2xy) - \frac{m}{2\pi} \ln a^2 \quad (2)$$

Since the logarithm of a complex quantity, $\zeta = |\zeta|e^{i\theta}$, is $\ln \zeta = \ln |\zeta| + i\theta$, for the imaginary part of Eq.(2), we have

$$\psi = \frac{m}{2\pi} \tan^{-1} \left(\frac{2xy}{x^2 - y^2 + a^2} \right) \quad (3)$$

since here $\theta = \tan^{-1} \left(\frac{2xy}{x^2 - y^2 + a^2} \right)$.

We obtain the x and y -components of the velocity, u_x and u_y respectively from

$$u_x = \frac{\partial \psi}{\partial y} = \frac{m}{\pi} \frac{x(x^2 + y^2 + a^2)}{a^4 + 2a^2x^2 - 2a^2y^2 + x^4 + 2x^2y^2 + y^4} \quad (4)$$

and

$$u_y = -\frac{\partial \psi}{\partial x} = \frac{m}{\pi} \frac{y(x^2 + y^2 - a^2)}{a^4 + 2a^2x^2 - 2a^2y^2 + x^4 + 2x^2y^2 + y^4} \quad (5)$$

To determine the velocity along the horizontal plane, we solve for u_x and u_y at $y = 0$. Accordingly,

$$u_{x,p} = \frac{m}{\pi} \frac{x(x^2 + a^2)}{a^4 + 2a^2x^2 + x^4} = \frac{m}{\pi} \frac{x}{x^2 + a^2} \quad (6)$$

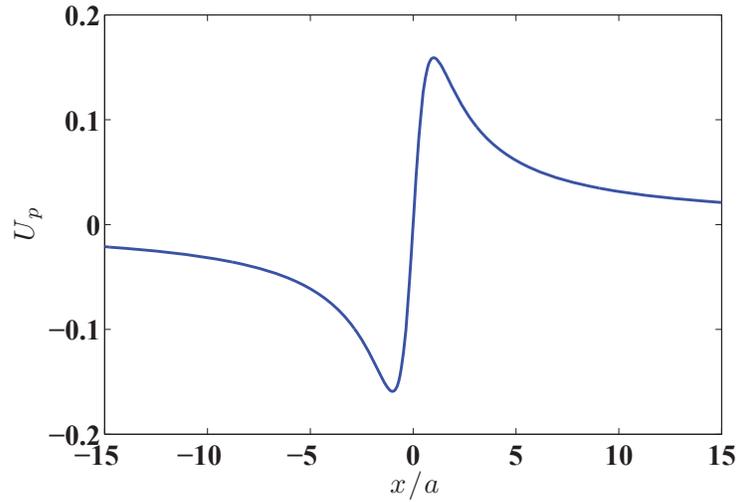
whereas

$$u_{y,p} = 0 \quad (7)$$

and hence the condition of no penetration along the plane is satisfied.

We can normalize this result to obtain

$$U_p \equiv \frac{u_{x,p}a}{m} = \frac{1}{\pi} \frac{\frac{x}{a}}{1 + \frac{x^2}{a^2}} \quad (8)$$

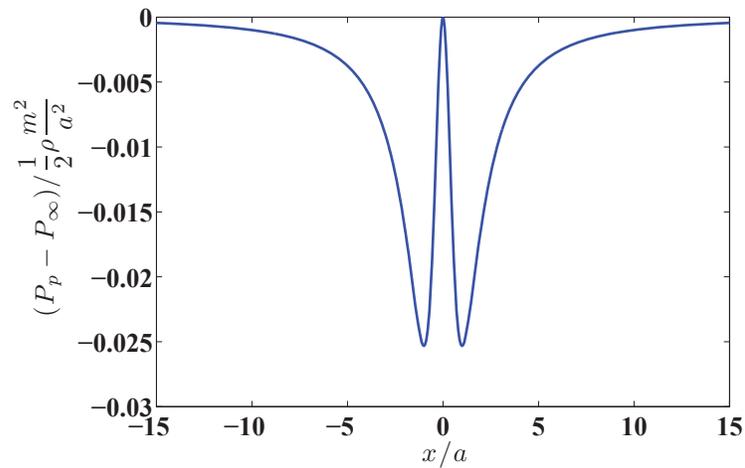


The pressure along the plane is given by the Bernoulli equation $P_\infty = P_p - \frac{1}{2}\rho u_{x,p}^2$, where P_∞ is taken to be given and is the pressure very far away from the plane where the velocity is essentially zero, but it also happens to equal the pressure at the stagnation point. So the pressure at the plane is

$$P_p = P_\infty - \frac{1}{2}\rho u_{x,p}^2 = P_\infty - \frac{1}{2}\rho \frac{m^2}{\pi^2} \frac{x^2}{(x^2 + a^2)^2} \quad (9)$$

This result can also be non-dimensionalized to

$$\frac{P_p - P_\infty}{\frac{1}{2}\rho \frac{m^2}{a^2}} = -\frac{1}{\pi^2} \frac{\frac{x^2}{a^2}}{\left(1 + \frac{x^2}{a^2}\right)^2} \quad (10)$$



The net upward (*i.e.* positive y -direction) force per unit depth, F' , on the plane is simply the integral of the gage pressure acting on it. Here we assume that the pressure on the bottom side of the plane is everywhere P_∞ .

$$F' = \int_{-\infty}^{\infty} -\Delta P \hat{n} dx = \int_{-\infty}^{\infty} -(P_p - P_\infty) dx = \int_{-\infty}^{\infty} \frac{1}{2} \rho \frac{m^2}{\pi^2} \frac{x^2}{(x^2 + a^2)^2} dx \quad (11)$$

The right hand side of Eq. (11) is

$$F' = \frac{1}{2} \rho \frac{m^2}{\pi^2} \left[\frac{1}{2a} \tan^{-1} \left(\frac{x}{a} \right) - \frac{x}{2(a^2 + x^2)} \right]_{-\infty}^{\infty} = \frac{1}{4a} \rho \frac{m^2}{\pi^2} \left[\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right] \quad (12)$$

So the net force acting per unit depth of the plane is

$$F' = \frac{1}{4\pi} \rho \frac{m^2}{a} \quad (13)$$

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