2.25 Fluid Mechanics

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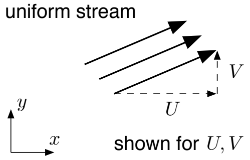
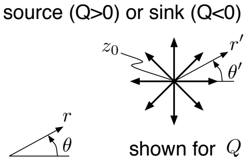
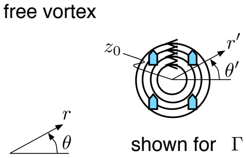
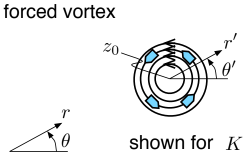
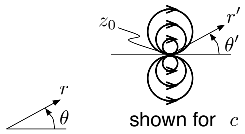
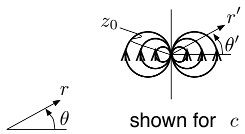
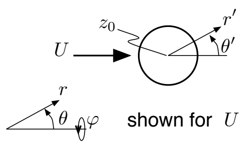
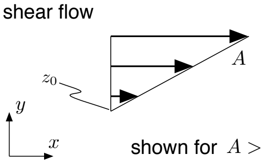
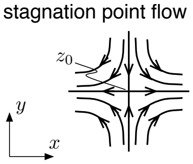
Stream Functions for planar flow (satisfy $\nabla \cdot \vec{v} = 0$)			
Planar flow: Cartesian $(x, y, \not{z})$	$v_x = \frac{\partial \psi}{\partial y}$	$v_y = -\frac{\partial \psi}{\partial x}$	$v_z = 0$
Planar flow: Cylindrical $(r, \theta, \not{z})$	$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$	$v_\theta = -\frac{\partial \psi}{\partial r}$	$v_z = 0$
Axisymmetric flow: Cylindrical $(r, \not{\theta}, z)$	$v_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}$	$v_\theta = 0$	$v_z = \frac{1}{r}\frac{\partial \psi}{\partial r}$
Axisymmetric flow: Spherical $(r, \theta, \not{\varphi})$	$v_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta}$	$v_\theta = -\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}$	$v_\varphi = 0$

Potential Functions ($\vec{v} = \nabla\phi$, requires $\nabla \times \vec{v} = 0, \nabla^2\phi = 0$)			
Cartesian coordinates (x, y, z)	$v_x = \frac{\partial \phi}{\partial x}$	$v_y = \frac{\partial \phi}{\partial y}$	$v_z = \frac{\partial \phi}{\partial z}$
Cylindrical coordinates (r, θ, z)	$v_r = \frac{\partial \phi}{\partial r}$	$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$	$v_z = \frac{\partial \phi}{\partial z}$
Spherical coordinates (r, θ, φ)	$v_r = \frac{\partial \phi}{\partial r}$	$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$	$v_\varphi = \frac{1}{r\,\sin\theta}\frac{\partial \phi}{\partial \varphi}$

uniform stream  shown for $U, V > 0$	$W(\mathrm{z}) = (U - iV)\mathrm{z}$ $\phi = Ux + Vy$ $\psi = -Vx + Uy$	$v_x = U$ $v_y = V$
source (Q>0) or sink (Q<0)  shown for $Q > 0$	$W(\mathrm{z}) = \frac{Q}{2\pi}\ln(\mathrm{z} - \mathrm{z}_0)$ $\phi = \frac{Q}{2\pi}\ln\, r'$ $\psi = \frac{Q}{2\pi}\theta'$	$v_r = \frac{Q}{2\pi}\frac{1}{r'}$ $v_\theta = 0$
free vortex  shown for $\Gamma > 0$	$W(\mathrm{z}) = \frac{-i\Gamma}{2\pi}\ln(\mathrm{z} - \mathrm{z}_0)$ $\phi = \frac{\Gamma}{2\pi}\theta'$ $\psi = -\frac{\Gamma}{2\pi}\ln\, r'$	$v_r = 0$ $v_\theta = \frac{\Gamma}{2\pi}\frac{1}{r'}$
forced vortex  shown for $K > 0$	$W(\mathrm{z}) = \nexists$ $\phi = \nexists$ $\psi = -\frac{Kr'^2}{2}$	$v_r = 0$ $v_\theta = Kr'$

doublet (x-orientation)  shown for $c > 0$	$W(\mathrm{z}) = \frac{c}{\mathrm{z}-\mathrm{z}_0}$ $\phi = \frac{c\,\cos\theta'}{r'}$ $\psi = -\frac{c\,\sin\theta'}{r'}$	$v_r = -\frac{c\,\cos\theta'}{r'^2}$ $v_\theta = -\frac{c\,\sin\theta'}{r'^2}$
doublet (y-orientation)  shown for $c > 0$	$W(\mathrm{z}) = \frac{ic}{\mathrm{z}-\mathrm{z}_0}$ $\phi = \frac{c\,\sin\theta'}{r'}$ $\psi = \frac{c\,\cos\theta'}{r'}$	$v_r = -\frac{c\,\sin\theta'}{r'^2}$ $v_\theta = \frac{c\,\cos\theta'}{r'^2}$
sphere (axisymmetric flow)  shown for $U > 0$	$W(\mathrm{z}) = \phi + i\psi$ $\phi = U\cos\theta'\left(r' + \frac{R^3}{2r'^2}\right)$ $\psi = \frac{1}{2}U\sin^2\theta'\left(r'^2 - \frac{R^3}{r'}\right)$	$v_r = U\cos\theta'\left(1 - \frac{R^3}{r'^3}\right)$ $v_\theta = -U\sin\theta'\left(1 + \frac{R^3}{2r'^3}\right)$ $v_\varphi = 0$
shear flow  shown for $A > 0$	$W(\mathrm{z}) = \nexists$ $\phi = \nexists$ $\psi = Ay'^2$	$v_x = 2Ay'$ $v_y = 0$ $v_z = 0$
stagnation point flow  shown for $A > 0$	$W(\mathrm{z}) = \frac{1}{2}A(\mathrm{z}-\mathrm{z}_0)^2$ $\phi = \frac{1}{2}A(x'^2 - y'^2)$ $\psi = Ax'y'$	$v_x = Ax'$ $v_y = -Ay'$ $v_z = 0$

Notes:

$\mathrm{z} = x + iy$ $\mathrm{z}_0 = x_0 + iy_0$ $0 \le \theta < 2\pi =$	$r' = \left[(x-x_0)^2 + (y-y_0)^2\right]^{\frac{1}{2}}$ $\theta' = \tan^{-1}\left(\frac{y-y_0}{x-x_0}\right)$	$W(\mathrm{z}) = \phi + i\psi$ $\frac{dW}{d\mathrm{z}} = v_x - iv_y$ $\frac{dW}{d\mathrm{z}} = (v_r - iv_\theta)e^{-i\theta}$
$v_x = v_r\cos\theta - v_\theta\sin\theta$ $v_y = v_r\sin\theta + v_\theta\cos\theta$	$v_r = v_x\cos\theta + v_y\sin\theta$ $v_\theta = -v_x\sin\theta + v_y\cos\theta$	