

Viscous Decay of a Free Vortex

This question revisits the model of a hurricane or tornado consisted in the first Quiz. We developed a model of a strongly rotating system that consisted of a rigid body motion for $r \leq R$ and an ideal vortex for $r \geq R$ and we calculated the distribution of viscous stress in the outer region of the flow. In a real tornado or hurricane or other vortical flow, viscous effects eventually cause the vortex to ‘spin down’ and decay away (unless of course the vortex is “fed” with energy – for example by natural convection over warm water).

To analyze this problem consider the following physical picture; a very long cylindrical rod of radius R dipped into a semi-infinite container of Newtonian fluid. The rod initially rotates at a constant angular velocity Ω and establishes a unidirectional flow $v_\theta(r,t)$ in the fluid for $r \geq R$.

- (a) Show that the steady-state velocity profile in the fluid is the same as that expected in an ideal vortex (for $r > R$) and by calculating the circulation for closed paths that either exclude or include the rod, show that the circulation and velocity is identical to that obtained in an axisymmetric free vortex of strength $\Gamma = 2\pi\Omega R^2$.
- (b) Find the torque (per unit length) exerted on this rod by the fluid.
- (c) Now imagine that the radius of the rod is decreased to zero and the rotation rate is correspondingly increased in such a way that the product $\Gamma = 2\pi\Omega R^2$ remains constant. In the limit $R \rightarrow 0$ we obtain the ideal line vortex discussed in class. At time $t = 0$, this limiting cylinder instantaneously stops rotating. The line vortex in the fluid will slowly decay away due to viscous dissipation of energy, and we postulate that the transient velocity field will remain of the form $v_\theta(r,t)$.

Write out and simplify (BUT DO NOT ATTEMPT TO SOLVE YET) the appropriate components of the Navier-Stokes equation and use the initial condition (of fixed circulation) to non-dimensionalize the velocity using a new variable $g = rv_\theta/(\Gamma/2\pi)$. Write down the initial condition and boundary conditions of this flow.

⇒ You may assume that the pressure field remains independent of θ at all times (as was true in the steady flow case).

- (c) Just as in the Rayleigh problem studied in class, there are no natural scales for r and t in this time-dependent viscous decay problem. Hence the solution will be a similarity solution in a dimensionless similarity variable $\eta = r/\sqrt{vt}$. This allows the partial differential equation you obtained above to be reduced to an ordinary differential equation and then solved. Use this given form of the similarity variable and substitute it into the θ component of the governing equation. Show that the equation reduces to an ordinary differential equation with appropriate collapse of the initial and boundary conditions.

(d) Show (by direct substitution) that the partial differential equation is satisfied by the following solution:

$$\frac{rV_\theta(r,t)}{\Gamma/2\pi} = \left[1 - \exp(-r^2/4t)\right] \quad \text{or equivalently} \quad g(\eta) = \left[1 - \exp(-\eta^2)\right] \quad (3.2)$$

Check that this solution also satisfies all of your initial and boundary conditions.

(d) Find an expression for the time-varying location of the maximum velocity in the flow, Also find an expression for the vorticity field. On side-by-side plots, sketch the velocity field and vorticity field for $t < 0$ and the evolution of these profiles for various times $t \geq 0$ (or you can make plots in MATLAB® and display them using the subplot(i,j).m command.

For more on this see: <http://www.spc.noaa.gov/faq/tornado/#The Basics>

The following expressions (in cylindrical polar coordinates) may be useful:

Conservation of Mass:
$$\frac{1}{r} \frac{\partial(rV_r)}{\partial r} + \frac{1}{r} \frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} = 0$$

The components of vorticity

$$\omega_r = \frac{1}{r} \frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z} \quad \omega_\theta = \frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r} \quad \omega_z = \frac{1}{r} \frac{\partial(rV_\theta)}{\partial r} - \frac{1}{r} \frac{\partial V_r}{\partial \theta}$$

The shear rate ($\dot{\gamma}$) on a plane with normal in the r direction;
$$\dot{\gamma}_{r\theta} = r \frac{\partial}{\partial r} \left(\frac{V_\theta}{r} \right) + \frac{1}{r} \frac{\partial V_r}{\partial \theta}$$

The Navier stokes equations in the r - θ plane:

$$\rho \left(\frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r} \frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r} + V_z \frac{\partial V_r}{\partial z} \right) = -\frac{\partial p}{\partial r} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial(rV_r)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V_r}{\partial \theta^2} + \frac{\partial^2 V_r}{\partial z^2} - \frac{2}{r^2} \frac{\partial V_\theta}{\partial \theta} \right] + \rho g_r$$

$$\rho \left(\frac{\partial V_\theta}{\partial t} + V_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} \frac{\partial V_\theta}{\partial \theta} + \frac{V_r V_\theta}{r} + V_z \frac{\partial V_\theta}{\partial z} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial(rV_\theta)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V_\theta}{\partial \theta^2} + \frac{\partial^2 V_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial V_r}{\partial \theta} \right] + \rho g_\theta$$

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2.25 Advanced Fluid Mechanics
Fall 2013

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