MIT Department of Mechanical Engineering

2.25 Advanced Fluid Mechanics

Stokes Second Problem ATP

Stokes apparently had many problems. This Second Problem is identical to the First Problem, except that we replace (2) with $u(y = 0, t) = Ucos(\omega t)$ — the plate now oscillates. Note that we are interested only in the steady periodic solution: u behaves as $cos(\omega t + \Phi^u)$ in time, where the phase Φ^u is independent of t. (The initial condition (4) is thus irrelevant —it washes out.)

In the steady-periodic state the wall shear stress will be of the form

$$\tau_W = CU^{\alpha_1}\rho^{\alpha_2}\mu^{\alpha_3}\omega^{\alpha_4}cos(\omega t + \Phi^\tau), \tag{1}$$

where the phase Φ^τ is independent of t and C is a non-dimensional constant. Find the exponents α_1, α_2, α_3 and α_4 by dimensional analysis.

Hint: (one approach): See Hint for Stokes' First Problem; make good use of the steady-periodic form of the solution.

Solution:

Let's start by non dimesionalizing the equations. Now write $u^* = \frac{u}{U}$; thus divided by (U).

$$\frac{\partial u^*}{\partial t} = \nu \frac{\partial^2 u^*}{\partial y^2} \qquad 0 < y < \infty, \tag{2}$$

$$u^*(y = 0, t) = cos(\omega t), \tag{3}$$

$$u^*(y \to \infty, t) \to 0, \tag{4}$$

$$u^*(y, t = 0) = 0, \tag{5}$$

and hence,

$$u^* = f(y, t, \nu, \omega), \tag{6}$$

(notice that no mass appears in the equations) so,

$$\Pi_1 = u^*, \tag{7}$$

$$\Pi_2 = ty^{\alpha 2}\omega^{\beta_2}, \tag{8}$$

$$\Pi_2 = \omega t, \tag{9}$$

$$\Pi_3^a = \nu y^{\alpha 3}\omega^{\beta_3}, \tag{10}$$

Solving the system of equations,

$$\alpha_3 = -2, \qquad \beta_3 = -1, \tag{11}$$

then,

$$\Pi_3^a = \frac{\nu}{y^2\omega}, \tag{12}$$

or,

$$\Pi_3 = \frac{y}{\sqrt{\frac{\nu}{\omega}}}. \tag{13}$$

Then, reexpressing the original function in terms of the non-dimensional parameters,

$$\Pi_1 = f_*(\Pi_2, \Pi_3), \tag{14}$$

or,

$$\frac{u}{U} = f_{**}(\frac{y}{\sqrt{\frac{\nu}{\omega}}}, \omega t), \tag{15}$$

Now, for the steady periodic behaviour $\frac{u}{U}$ must be 'sinusoidal' in time, so

$$\frac{u}{U} = A\left(\frac{y}{\sqrt{\frac{\nu}{\omega}}}\right)\cos\left(\omega t + \Phi\left(\frac{y}{\sqrt{\frac{\nu}{\omega}}}\right)\right) \tag{16}$$

where,

$$A(0) = 1 \qquad and \qquad \Phi(0) = 0. \tag{17}$$

Furthermore,

$$\tau_W = -\mu\frac{\partial u}{\partial y}|_{y=0} \tag{18}$$

$$\tau_W = -\mu U\left(A'(0)\frac{1}{\sqrt{\frac{\nu}{\omega}}})\cos(\omega t + \Phi(0)) - A(0)\frac{1}{\sqrt{\frac{\nu}{\omega}}})\sin(\omega t + \Phi(0))\Phi'(0)\right) \tag{19}$$

$$\tau_W = -\mu U\frac{1}{\sqrt{\frac{\nu}{\omega}}}\left(A'(0)\cos(\omega t + \Phi(0)) - \Phi'(0)\sin(\omega t + \Phi(0))\right). \tag{20}$$

Finally, the last equation can be reexpressed as:

$$\tau_W = -\mu U\frac{1}{\sqrt{\frac{\nu}{\omega}}}\left(A'(0)^2 + \Phi'(0)^2\right)^{0.5}\cos(\omega t + \Phi_(0)^\tau). \tag{21}$$

where, ${\Phi_0}^\tau = \arctan\frac{\Phi'(0)}{A'(0)}$; and $A'(0)$ and $\Phi'(0)$ are 'universal constants'.

□

Problem Solution by MC, Fall 2008

