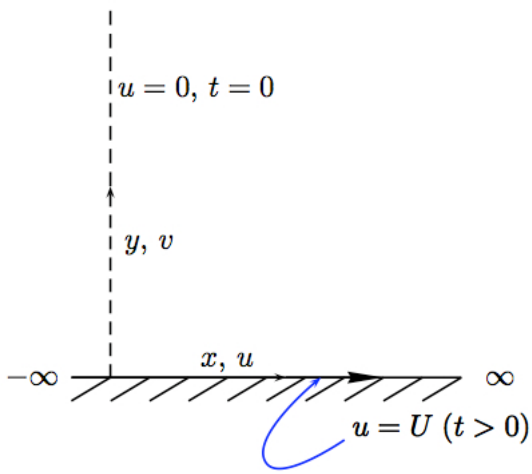
MIT Department of Mechanical Engineering

2.25 Advanced Fluid Mechanics

Stokes First Problem ATP



Consider Stokes' First Problem: impulsive start of a flat plate beneath a semi-infinite layer of initially quiescent incompressible fluid. The governing equations (presuming parallel flow — no instabilities) for $u(y,t)$ are:

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2}, \qquad 0 < y < \infty \ , \tag{1}$$

$$u(y = 0, t) = U \ , \tag{2}$$

$$u(y \to \infty, t) \to 0 \ , \tag{3}$$

$$u(y, t = 0) = 0 \ . \tag{4}$$

The shear stress at the wall is then given by

$$\tau_W(t) = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} \ . \tag{5}$$

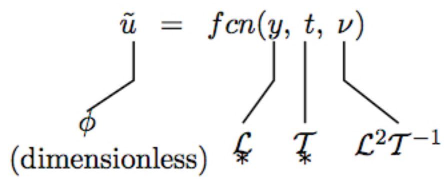
Here ρ is the density and μ is the dynamic viscosity. The shear stress at the wall will be of the form

$$\tau_W = C U^{\alpha_1} \rho^{\alpha_2} \mu^{\alpha_3} t^{\alpha_4} \ , \tag{6}$$

where C is a non-dimensional constant. Find the exponents α_1, α_2, α_3, and α_4 by dimensional analysis.

Hint (one approach): Write the equations in terms of u/U; apply Buckingham Pi with as few variables as possible; apply the chain rule.

Solution:



where $\tilde{u}$ is dimensionless; y has units of length, $\mathcal{L}$; y has units of length, $\mathcal{L}$; t has units of time, $\mathcal{T}$, and ν is given in $\mathcal{L}^2\mathcal{T}^{-1}$. Then, there are three remaining variables and two remaining dimensions; therefore there is one more dimensional group.

So, $\Pi_1 = \tilde{u}$ (or any multiple), and $\Pi_2 = \nu y^{\alpha_2} t^{\beta_2}$. Now, choosing α_2, β_2 such that,

$$2 + \alpha_2 = 0 \ , \tag{7}$$

$$-1 + \beta_2 = 0 \ , \tag{8}$$

$$\alpha_2 = -2 \quad , \quad \beta_2 = 1 \ , \tag{9}$$

Then,

$$\Pi_2 = \nu t / y^2 \tag{10}$$

or

$$\Pi_1 = fcn_*^0(\Pi_2^0) \tag{11}$$

where Π_2^0 can be replaced with any function of Π_2^0; choose (convention) $\Pi_2 = (\Pi_2^0)^{-1/2} = y/\sqrt{\nu t}$

$$\Pi_1 = fcn_*(\Pi_2) \ , \tag{12}$$

or

$$\frac{u}{U} = fcn_*(y/\sqrt{\nu t}) = f_I(y/\sqrt{\nu t}) \tag{13}$$

furthermore, using the chain rule,

$$\tau_W = -\mu \frac{\partial u}{\partial y}|_{y=o} = -\mu f_I'(0) \frac{U}{\sqrt{\nu t}}, \tag{14}$$

where $f_I'(0)$ is a 'universal constant'.

Now, if we define

$$\delta_\epsilon = y \qquad such \quad that \qquad \frac{u(\delta_\epsilon)}{U} = \epsilon, \tag{15}$$

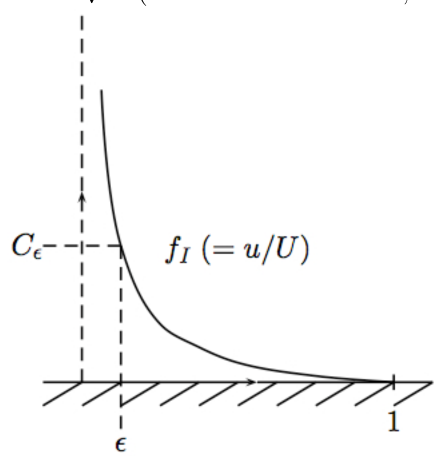
then,

$$f_I(\frac{\delta_\epsilon}{\sqrt{\nu t}}) = \epsilon \tag{16}$$

$$\frac{\delta_\epsilon}{\sqrt{\nu t}} = C_\epsilon, \tag{17}$$

$$\delta_\epsilon = C_\epsilon \sqrt{\nu t}. \tag{18}$$

Then, the effect of the wall penetrates as $\sqrt{\nu t}$ (shear accelerates fluid, decreases stress, ...).



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Problem Solution by MC, Fall 2008

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