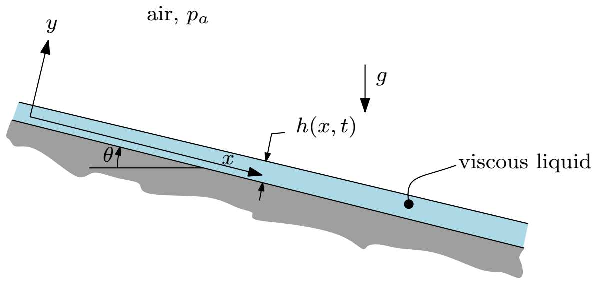


A rigid plane surface is inclined at an angle θ relative to the horizontal and wetted by a thin layer of highly viscous liquid which begins to flow down the incline.

1. Show that if the flow is two-dimensional and in the inertia-free limit, and if the angle of the inclination is not too small, the local thickness $h(x,t)$ of the liquid layer obeys the equation
$$\frac{\partial h}{\partial t} + c\frac{\partial h}{\partial x} = 0$$
where
$$c = \frac{\rho g h^2}{\mu}\sin\theta$$

2. Demonstrate that the result of (a) implies that in a region where h decreases in the flow direction, the angle of the free surface relative to the inclined plane will steepen as the fluid flows down the incline, while in a region where h increases in the flow direction, the reverse is true. Does this explain something about what happens to slow-drying paint when it is applied to an inclined surface?

3. Considering the result of (b) above, do you think that the steady-state solutions of the previous problems would ever apply in practice? Discuss.

1. Assumptions:

   - $\mathrm{Re}_H \frac{H}{L} \ll 1$
   - two dimensional flow
   - THIN layer $\Rightarrow \frac{H}{L} = \frac{\text{characteristic height}}{\text{characteristic length}} \ll 1$

   Unknown: $h(x,t)$?

   (1) Choose relevant scales: (* denotes dimensionless variables)

$$x^* = \frac{x}{L} \qquad v_x^* = \frac{v_x}{U} \qquad t^* = \frac{tU}{L}$$
$$y^* = \frac{y}{H} \qquad v_y^* = \frac{v_y}{V} \qquad p^* = \frac{p}{\mathcal{P}}$$

   where $\mathcal{P}$ is an unknown pressure scale.

   (2) Non-dimensionalize continuity:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$
$$\frac{U}{L}\frac{\partial v_x^*}{\partial x^*} + \frac{V}{L}\frac{\partial v_y^*}{\partial y^*} = 0$$

   Since dimensionless variables are assumed to be of the same order ($\mathcal{O}(1)$),

$$\frac{U}{L} \sim \frac{V}{H}$$
$$\Rightarrow V = \frac{H}{L}U \qquad \text{where} \quad \frac{H}{L} \ll 1$$

   (3) Non-dimensionalize Navier-Stokes:

   <u>x-direction:</u>

$$\rho\frac{U^2}{L}\left(\frac{\partial v_x^*}{\partial t^*} + v_x^*\frac{\partial v_x^*}{\partial x^*} + v_y^*\frac{\partial v_x^*}{\partial y^*}\right) = -\frac{\mathcal{P}}{L}\frac{\partial p^*}{\partial x^*} + \rho g \sin\theta + \mu\left(\frac{U}{L^2}\frac{\partial^2 v_x^*}{\partial x^{*2}} + \frac{U}{H^2}\frac{\partial^2 v_x^*}{\partial y^{*2}}\right)$$

   Divide through by $\frac{\mu U}{H^2}$:

$$\underbrace{\frac{\rho U H^2}{\mu L}}_{\mathrm{Re}_H \frac{H}{L}=\text{small}} \cancel{\left(\frac{\partial v_x^*}{\partial t^*} + v_x^*\frac{\partial v_x^*}{\partial x^*} + v_y^*\frac{\partial v_x^*}{\partial y^*}\right)} = -\frac{\mathcal{P}H^2}{\mu U L}\frac{\partial p^*}{\partial x^*} + \frac{\rho g H^2 \sin\theta}{\mu U} + \underbrace{\left(\frac{H}{L}\right)^2}_{\text{small}}\cancel{\frac{\partial^2 v_x^*}{\partial x^{*2}}} + \frac{\partial^2 v_x^*}{\partial y^{*2}}$$

   <u>y-direction:</u>

$$\rho\frac{U^2}{L}\frac{H}{L}\left(\frac{\partial v_y^*}{\partial t^*} + v_x^*\frac{\partial v_y^*}{\partial x^*} + v_y^*\frac{\partial v_y^*}{\partial y^*}\right) = -\frac{\mathcal{P}}{H}\frac{\partial p^*}{\partial y^*} - \rho g\cos\theta + \mu\frac{H}{L}\left(\frac{U}{L^2}\frac{\partial^2 v_y^*}{\partial x^{*2}} + \frac{U}{H^2}\frac{\partial^2 v_y^*}{\partial y^{*2}}\right)$$

   Divide through by $\frac{\mu U}{H^2}$:

$$\underbrace{\frac{\rho U H^2}{\mu L}}_{\mathrm{Re}_H \left(\frac{H}{L}\right)^2=\text{small}} \cancel{\left(\frac{\partial v_y^*}{\partial t^*} + v_x^*\frac{\partial v_y^*}{\partial x^*} + v_y^*\frac{\partial v_y^*}{\partial y^*}\right)} = -\frac{\mathcal{P}H}{\mu U}\frac{\partial p^*}{\partial y^*} - \frac{\rho g H^2\cos\theta}{\mu U} + \underbrace{\left(\frac{H}{L}\right)^3}_{\text{small}}\cancel{\frac{\partial^2 v_y^*}{\partial x^{*2}}} + \underbrace{\frac{H}{L}}_{\text{small}}\cancel{\frac{\partial^2 v_y^*}{\partial y^{*2}}}$$

Corrected part starts from here: Note that it is generally better and also more precise not to do scaling for pressure before writing the simplified Navier-Stokes equation in both x and y components (long and short scales in the problem...for a problem in cylindrical coordinates it may turn into r and z for example). One should do the scaling for pressure after simplifying the NSE and based on the order of the left terms. This is especially very important in the cases that we have a free surface at one end of the small scale coordinate ($y-coordinate$ in this problem). A common mistake is to use the scaling for pressure that we had in the lubrication problem for bearings. Whenever there is a free surface at one end of the smaller scale which its height is a function of time, i.e. $h(x,t)$ it is a good idea to start from the simplified N.S.E for the small scale component (here y component):

y-direction:

$$\underbrace{\frac{\rho U H^2}{\mu L}}_{\mathrm{Re}_H\left(\frac{H}{L}\right)^2=\text{small}}\left(\frac{\partial v_y^*}{\partial t^*}+v_x^*\frac{\partial v_y^*}{\partial x^*}+v_y^*\frac{\partial v_y^*}{\partial y^*}\right)=-\frac{\mathcal{P}H}{\mu U}\frac{\partial p^*}{\partial y^*}-\frac{\rho g H^2\cos\theta}{\mu U}+\underbrace{\left(\frac{H}{L}\right)^3}_{\text{small}}\frac{\partial^2 v_y^*}{\partial x^{*2}}+\underbrace{\frac{H}{L}}_{\text{small}}\frac{\partial^2 v_y^*}{\partial y^{*2}}$$

Now if one brings it back to the dimensional form it will be:

y-direction:

$$\boxed{0=-\frac{\partial p}{\partial y}-\rho g cos\theta.}$$

We also have the knowledge that surface tension effects are not important so we already know that the pressure at the interface should be equal to p_a due to the fact that across an interface the normal stress (pressure here) will not change when surface tension effects can be neglected. This gives us a boundary condition for pressure, $p(at\ \ y=h(x,t))=p_a$ which is valid for any arbitrary x. Using the mentioned boundary condition and integrating the simplified y-component of NSE along y we obtain the following expression for pressure:

$$\boxed{p=p_a+\rho g cos\theta\left(h(x,t)-y\right)}$$

Now notice that the mentioned relationship is valid for any arbitrary x. This is true because when we integrate the NSE along y we reach to P_a at the interface and we assume with P_a does not change significantly with x (i.e. density of air is negligible). If the density of the other media (air in this example) is not negligible then we should take the hydrostatic changes of p_a into account (see Shapiro 6.13 for such an example).Now back to the fact that in this problem this is valid for all x, thus:

$$p\left(x,y\right)=p_a+\rho g cos\theta\left(h(x,t)-y\right)\Rightarrow\boxed{\frac{\partial p}{\partial x}=\rho g cos\theta\frac{\partial h}{\partial x}\sim\rho g cos\theta\frac{h}{L}<<\rho g sin\theta}$$

Now if we look at the simplified x-component of NSE in the dimensional form and use the fact that $\partial p/\partial x<<\rho g sin\theta$ then we will have: (we are assuming here that θ is not vanishingly small)

$$\boxed{0=\rho g\sin\theta+\mu\frac{\partial^2 v_x}{\partial y^2}}$$

Please note that here the gravity is running the flow and the viscous terms are resisting against it. In free surface lubrication problems usually there is a way to determine the pressure gradient term $\partial p/\partial x$ and then usually you end up finding that the main terms driving the flow is gravity, or maybe centripetal acceleration in the rotational frames and the resisting terms are of viscous nature like the general case.
The rest of solution will be as it was before(please also check my explanations for conservation of mass written few lines below): Going back to the dimensional form,

$$0 = \rho g \sin\theta + \mu \frac{\partial^2 v_x}{\partial y^2} \tag{1}$$

(4) Solve for v_x by integrating both sides of Eq. (1):

$$\int \frac{\partial^2 v_x}{\partial y^2} dy = -\int \frac{\rho g \sin\theta}{\mu} dy$$

$$\frac{\partial^2 v_x}{\partial y^2} = -\frac{\rho g \sin\theta}{\mu} y + C_1$$

Using B.C. that $\frac{\partial v_x}{\partial y} = 0$ at $y = h$ (free surface),

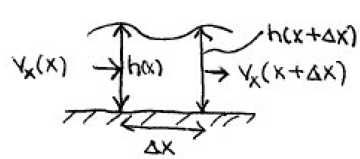
$$C_1 = \frac{\rho g h \sin\theta}{\mu}$$

$$\Rightarrow \frac{\partial^2 v_x}{\partial y^2} = -\frac{\rho g \sin\theta}{\mu}(y - h)$$

Integrating again and using no-slip B.C. ($v_x = 0$ at $y = 0$):

$$v_x = \frac{\rho g \sin\theta}{\mu}\left(hy - \frac{1}{2}y^2\right) \tag{2}$$

(5) Use mass conservation to obtain a single evolution equation for $h(x, t)$.
This is a very strong tool when you have a free surface problem in which h is a function of time as well as position i.e. $h(x, t)$ to obtain an evolution equation for $h(x, t)$ Consider the following control volume in the limit of $\Delta x \to 0$:



$$\frac{d}{dt}\int_{CV} \rho\, dV + \rho \int_{CS} (\mathbf{v} - \cancel{\mathbf{v}_c}) \cdot \hat{\mathbf{n}} dA = 0$$

$$\lim_{\Delta x \to 0} \frac{\cancel{\rho}\frac{d}{dt}(h\cancel{\Delta x}) + \cancel{\rho}\int_0^h v_x dy}{\cancel{\Delta x}} = 0$$

$$\Rightarrow \boxed{\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\int_0^h v_x dy\right) = \frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = 0} \tag{3}$$

in which Q is the volumetric flow rate per unit depth and is defined as

$$Q \equiv \int_0^{h(t)} V_x(y).$$

The above equation can also be derived by combining the kinematic boundary condition, $\frac{\partial h}{\partial t} + v_x \frac{\partial h}{\partial x} = v_y|_{y=h(x)}$, with conservation of mass.

Note that this form is true only for Cartesian Coordinates. It is easy to derive a similar equation for cylindrical coordinates and I leave it as an excercise for you. Your final answer in cylindrical coordinates will be:

$$\boxed{\frac{\partial h}{\partial t} + \frac{1}{r}\frac{\partial (rQ)}{\partial r} = 0}$$

(6) Combine Eq. (2) with Eq. (3):

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_0^h \frac{\rho g \sin\theta}{\mu}(hy - \frac{1}{2}y^2)dy = 0$$

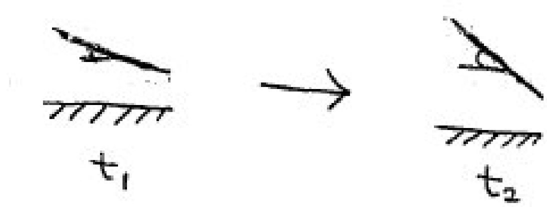
$$\frac{\partial h}{\partial t} + \frac{\rho g \sin\theta}{\mu}\frac{\partial}{\partial x}\left(\frac{h^3}{3}\right) = 0$$

$$\boxed{\frac{\partial h}{\partial t} + \underbrace{\left(\frac{\rho g \sin\theta}{\mu}h^2\right)}_{c}\frac{\partial h}{\partial x} = 0} \tag{4}$$

Eq. (4) is a nonlinear wave equation with a solution of the form $h = f(x - ct)$, where c is the wave speed.

2. Since $\frac{\rho g \sin\theta}{\mu}h^2 \geq 0$, $\frac{\partial h}{\partial t}$ and $\frac{\partial h}{\partial x}$ have opposite signs to satisfy Eq. (4).
Thus, where h is decreasing locally ($\frac{\partial h}{\partial x} < 0$), h increases in time ($\frac{\partial h}{\partial t} > 0$).

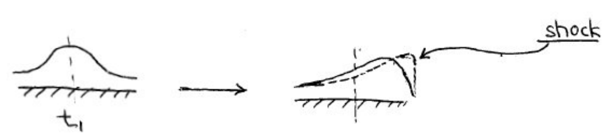
Angle of free surface steepens because points of larger h increase more rapidly ($c \sim h^2$) than points of lower h:



Where h is increasing locally ($\frac{\partial h}{\partial x} > 0$), h decreases in time ($\frac{\partial h}{\partial t} < 0$),

Angle of free surface flattens because points of larger h decrease more rapidly ($c \sim h^2$) than points of lower h:

⇒ In the case of slow-drying paint, when there is a bump, Eq. (4) dictates that the bump grows! However it never forms a shock because in reality, one has to consider effects of surface tension.



In practice, the solution to Eq. (4) fails (or goes unstable) in the case of a symmetric perturbation, as explained in (b). Thus, it is not very applicable unless one accounts or effects of surface tension and such.

However, when h is monotonically increasing ($\frac{\partial h}{\partial x} > 0$ everywhere) the solution to Eq. (4) is indeed stable since it predicts that h flattens in time.

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