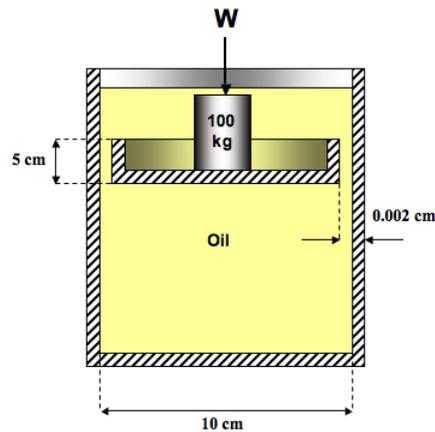


MIT Department of Mechanical Engineering
2.25 Advanced Fluid Mechanics

Problem 6.01

This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin



Oil is confined in a 10 [cm] diameter cylinder by a piston with a clearance of 0.0002 [cm]. The piston is 5 [cm] long, and the oil has a viscosity coefficient of 0.05 [kg/ms] and a density of 920 [kg/m³].

A total weight of 100 [kg] is applied to the piston. Estimate the leakage rate of oil past the piston, in *liters/day*. Justify any approximations you use in arriving at your estimate.

Solution:

Let's calculate the Reynolds number for this problem,

$$Re = \frac{\rho V H}{\mu} = \frac{920 * U * 0.00002}{0.05} = 0.368U, \quad (6.01a)$$

then, even assuming a velocity of $0.1[m/s]$, the Reynold's number is small, and the lubrication approximation ¹ (Low Reynold's number) can be used as an initial assumption ². Also, we'll assume a pressure driven flow because, from the flow geometry,

$$U_{piston} = \frac{Q_{oil}}{A_{piston}} = \frac{U_{Oil,avg} 2\pi R H}{\pi R^2} = 2U_{Oil,avg} \frac{H}{R}, \quad (6.01b)$$

is pretty small, and therefore the viscous flow that it creates.

From the N-S in cylindrical coordinates, the equation can be reduced to (using the low Reynolds approximation for Pressure driven flow),

$$0 = -\frac{1}{\mu} \frac{\partial p}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right), \quad (6.01c)$$

where the first term, pressure drop rate, is approximately a constant across the space between the cylinders (long cylinder/small gap approximation), then

$$0 = -K + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right). \quad (6.01d)$$

Furthermore, since the gap is small, $1 \gg h/R$, it is possible to approximate this problem using a local cartesian coordinate system, (see problem 6.4, Sonin and Shapiro)

$$0 = -K + \frac{\partial (v_z)^2}{\partial y^2}. \quad (6.01e)$$

then, integrating,

$$v_z = K \frac{y^2}{2} + C_1 y + C_2. \quad (6.01f)$$

Then, applying the boundary conditions,

$$v_z(y = 0) = 0, \quad (6.01g)$$

$$v_z(y = H) = 0, \quad (6.01h)$$

¹The lubrication approximation, $Re_h \frac{h}{L} \ll 1$ is really less restrictive, the Reynolds number can be large, as long as this combination of parameters gives a small number still, but since $\frac{h}{L} < 1$ having the Reynolds number small is more than enough.

²We'll verify this later.

where $H = R_2 - R_1 = 0.002[cm]$, and the origin of the reference coordinate system, $y = 0$, is located at $r = R_1$. Now, using the B.C., the constants can be obtained,

$$C_2 = 0, \quad (6.01i)$$

$$C_1 = -K \frac{H}{2}, \quad (6.01j)$$

then,

$$v_z = K \left(\frac{y^2}{2} - \frac{Hy}{2} \right). \quad (6.01k)$$

then, the resulting velocity profile is a Poiseuille Flow (notice that this could have been inferred since the beginning of the problem). Now, to obtain the flux, let's integrate this expression,

$$2\pi R \int_0^H v_z = 2\pi R \int_0^H K \left(\frac{y^2}{2} - \frac{Hy}{2} \right) dy. \quad (6.01l)$$

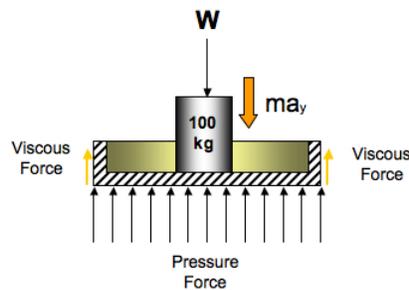
After integration,

$$Q = \pi R K \left(\frac{y^3}{3} - \frac{Hy^2}{2} \right) \Big|_0^H \quad (6.01m)$$

then, finally,

$$Q = -\frac{\pi R}{6} K H^3 = \frac{\pi R \Delta P}{6L\mu} H^3. \quad (6.01n)$$

Now, let's look at a force balance of the piston,

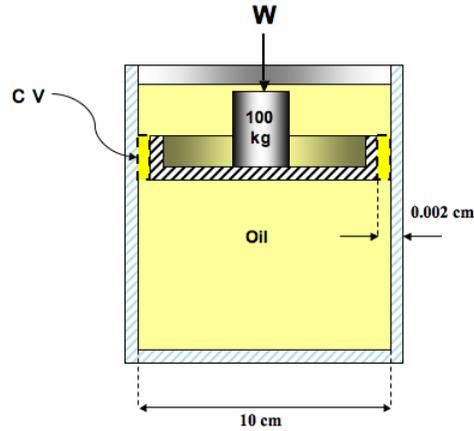


$$\sum F_y = -W + F_{viscous} + \Delta P \pi R^2 = (Mass_p) a_y. \quad (6.01o)$$

In this case the weight of the piston creates almost all the total pressure and viscous stresses can be neglected. To justify this, let's take a look at the viscous stresses for a second. The viscous force upwards is supplied

by the pressure force exerted on the liquid just beneath and entering the gap between the piston and the cylinder, and can't be higher than that (think about Momentum Conservation in a CV that encompasses the oil around the piston), then,

$$F_{viscous} \sim \Delta P A_{gap} \approx 2\pi R H \Delta P, \quad (6.01p)$$



on the other hand, the force coming from the pressure beneath the piston is

$$F_{Piston\ Face} = \Delta P \pi R^2, \quad (6.01q)$$

then, the viscous force is much more smaller than the pressure force beneath the piston and almost completely balances by itself the piston weight³. Then,

$$-W + \Delta P \pi R^2 = 0. \quad (6.01r)$$

Therefore, the pressure under the cylinder can be calculated as

$$P_{below} = \frac{W}{Area} + P_a = \frac{mg}{\pi(r)^2} + P_a = \frac{100 * 9.8[N]}{\pi(0.05)^2[m^2]} + P_a = \frac{39.2e4}{\pi}[N/m^2] + P_a \quad (6.01s)$$

Here, the change in pressure due to gravity was neglected because the distance is small to be important, more precisely, the total pressure change due to gravity is

$$P = \rho g H = 920 * 9.8 * 0.05[N/m^2] = 450[Pa], \quad (6.01t)$$

which is 1% of the pressure due to the weight, then

$$\frac{\partial P}{\partial z} = \frac{39.2e4}{\pi L}[N/m^2] \quad (6.01u)$$

³The acceleration is small because we have assumed a slow process as part of the low Reynolds assumption, to be checked later.

Now, substituting the given values,

$$Q = \frac{\pi(0.05[m])(39.2e4[Pa])}{6\pi(0.05[m])(0.05[kg/(ms)])} ((2e(-5))[m])^3 = 1.04e(-8)[m^3/s], \quad (6.01v)$$

Now transforming into *liters/hour*,

$$Q = 37[ml/hour] \quad (6.01w)$$

□

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