

### Unsteady Bernoulli Equation

In addition to understanding the effects of fluid acceleration in steady flow, we are also interested in impulsively started flows and transient flows. For such flows the Euler equation will be:

$$\rho \frac{\partial v_s}{\partial t} + \rho v_s \frac{\partial v_s}{\partial s} = -\frac{\partial P}{\partial s} + \rho g_s$$

We now integrate this along a streamline:

$$\int_1^2 \rho \frac{\partial v_s}{\partial t} ds + \int_1^2 \rho v_s \frac{\partial v_s}{\partial s} ds = -(P_2 - P_1) - \rho g(z_2 - z_1)$$

which can be simplified to:

$$\boxed{\int_1^2 \rho \frac{\partial v_s}{\partial t} ds + (P + \frac{1}{2} \rho v_s^2 + \rho g z)_2 - (P + \frac{1}{2} \rho v_s^2 + \rho g z)_1 = 0} \quad (1)$$

This is not a very useful result in general since  $\partial v_s / \partial t$  can change dramatically from one point to another; to use this in practice we need to be able to draw streamline shapes at each instant in time. It works especially for simple cases such as impulsively started confined flows where streamlines have the same shape at each instant and we are interested in time required to start the flow.

**Example: Flow out of a long pipe connected to a large reservoir (steady and transient starting stages)**

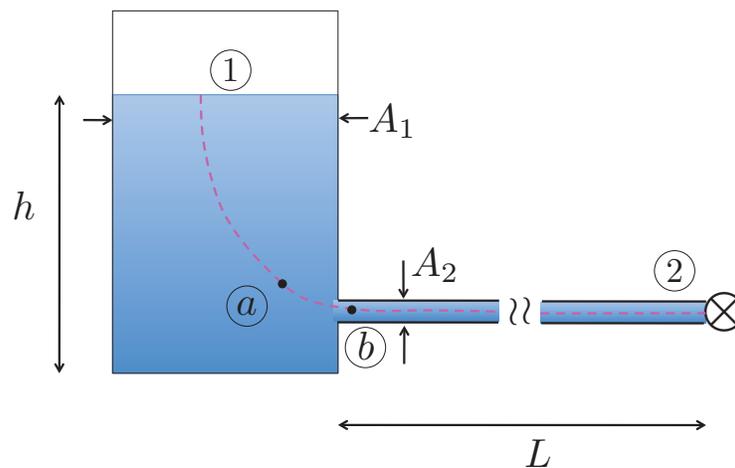


Figure 1: Discharge of water from a long pipe connected to a large fluid reservoir with cross section area  $A_1 \gg A_2$ . The problem approaches steady state when the valve has been open for a “long time” but is transient in the starting stage.

Consider the flow in the discharge of water through a long pipe connected to a big reservoir. If the area of the tank is much larger than the pipe cross section area (i.e.  $A_1 \gg A_2$ ) then the

solution for steady state case, in which the discharge valve has been open for a while, can be easily done by writing Bernoulli between points (1) and (2):

$$P_a + \frac{1}{2}\rho(0)^2 + \rho gh = P_a + \frac{1}{2}\rho v_2^2 + \rho g(0) \Rightarrow \boxed{v_2 = \sqrt{2gh}}$$

where  $v_1 A_1 = v_2 A_2 \Rightarrow v_1 \simeq 0$  because  $A_2 \ll A_1$ .

This result was known to Torricelli in the 1630. Now consider the analysis for a more general case which includes the starting time. Although the velocity is changing with time in the pipe during the transient stage one can easily conclude that conservation of mass says that velocity has to be constant at any instant along the length of the pipe and it just changes with time. Applying unsteady Bernoulli equation, as described in equation (1) will lead to:

$$\int_1^2 \rho \frac{\partial v_s}{\partial t} ds + (P_a + \frac{1}{2}\rho(v_2)^2 + \rho g(0)) - (P_a + \frac{1}{2}\rho(0)^2 + \rho gh) = 0 \quad (2)$$

Calculating an exact value for the first term on the left hand side is not an easy job but it is possible to break it into several terms:

$$\int_1^2 \rho \frac{\partial v_s}{\partial t} ds = \int_1^a \rho \frac{\partial v_s}{\partial t} ds + \int_a^b \rho \frac{\partial v_s}{\partial t} ds + \int_b^2 \rho \frac{\partial v_s}{\partial t} ds$$

If the reservoir area is much larger than the pipe area then the integral from (1) to (a) is negligible compared to the integral along the pipe length ((b) to (2)) because  $v_s$  in the tank is small, also knowing that the entry region is small compared to the length of the pipe we can easily neglect the integral from (a) to (b) compared to the corresponding integral over the pipe length. Thus the following estimate is an acceptable approximation for the unsteady term in the Bernoulli equation:

$$\int_1^2 \rho \frac{\partial v_s}{\partial t} ds \simeq \int_b^2 \rho \frac{\partial v_s}{\partial t} ds = \int_b^2 \rho \frac{\partial v_2}{\partial t} ds = \rho \frac{\partial v_2}{\partial t} L \quad (3)$$

Combining (2) and (3) will result in:

$$\rho \frac{\partial v_2}{\partial t} L + \frac{1}{2}\rho v_2^2 = \rho gh$$

It is worthy to mention that in this equation both  $v_2$  and  $h$  are in reality changing with time but for simplifying the analysis one can assume that the pressure head (i.e.  $h$  in the tank) remains almost unchanged during the transient starting stage (physically also it is right to assume that changes in  $h$  are almost negligible compared to other terms since the area of the tank is really large ( $A_1 \gg A_2$ ) and it takes a lot of fluid flow through the pipe to see changes in  $h$ ). Assuming a constant value of  $h$  the simplified equation will be:

$$\frac{dv_2}{dt} L + \frac{1}{2}v_2^2 = gh \rightarrow \frac{2L}{2gh - v_2^2/2} dv_2 = dt$$

integrating from  $t = 0$  and knowing that  $v_2 = 0$  at  $t = 0$  gives the following integral:

$$\int_0^{v_2} \frac{dv_2}{2gh - v_2^2/2} = \int_0^t \frac{dt}{2L} \rightarrow \frac{1}{2\sqrt{2gh}} \int_0^{v_2} \left( \frac{1}{\sqrt{2gh} + v_2} + \frac{1}{\sqrt{2gh} - v_2} \right) dv_2 = \int_0^t \frac{dt}{2L}$$

which can be simplified to:

$$v_2 = \sqrt{2gh} \tanh \left( \frac{\sqrt{2gh}}{L} t \right) \Rightarrow \boxed{v_2 = \sqrt{2gh} \tanh(t/\tau)} \quad (4)$$

where the characteristic time constant is  $\tau \equiv L(\sqrt{2gh})^{-1}$ .

The described relationship for the transient velocity and time is plotted in Figure 2.

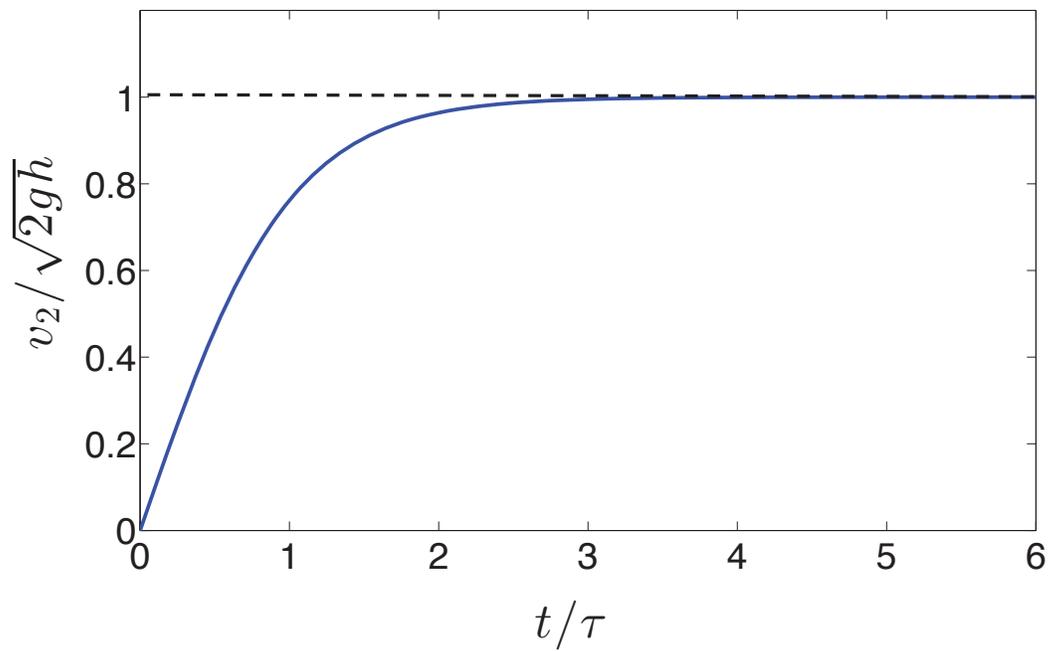


Figure 2: Velocity in the pipe as a function of time. The characteristic time constant is  $\tau \equiv L(\sqrt{2gh})^{-1}$

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