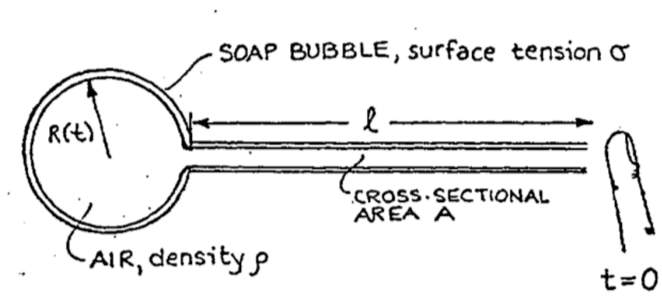
MIT Department of Mechanical Engineering
2.25 Advanced Fluid Mechanics

Problem 4.12

This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin

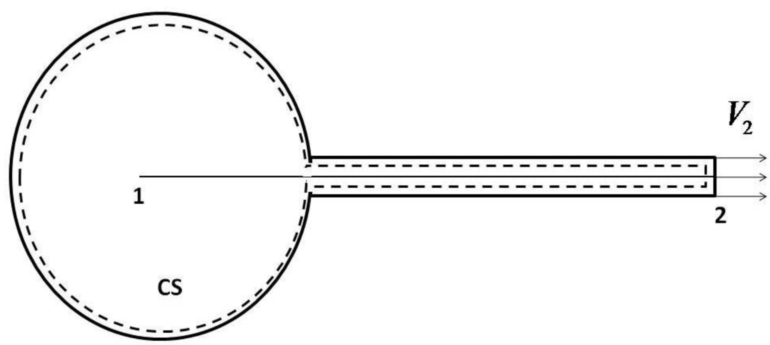


A soap bubble (surface tension σ) is attached to a narrow glass tube of the dimensions shown. The initial radius of the bubble is R_0. At $t = 0$, the end of the tube is abruptly opened.

a) Obtain a solution for $R(t)$, assuming that the flow is : (i) incompressible ans (ii) inviscid, that (iii) gravitational effects are negligible, and that (iv) the temporal acceleration term in Euler's equation is negligible (we are referring to the term involving the partial derviative of the velocity with time).

b) Derive a criterion for when assumption (iv) is satisfied.

Solution:



(a) To obtain an expression for R, we will write mass conservation across the moving control volume outlined in the figure above then since the flow is inviscid, incompressible and steady, we can write Bernoulli's equation (conservation of momentum) across a streamline from point 1 to point 2.

- Mass conservation

  Using form A for mass conservation on the moving control volume drawn above, we have

$$\frac{d(\int_{CV(t)}\rho\, dV)}{dt} + \int_{CS(t)} \rho(\mathbf{v} - \mathbf{v}_c).\mathbf{n}\, dA = 0 \tag{4.12a}$$

  The change in volume $\frac{dV}{dt}$ is the change in volume of the bubble

$$\frac{d(\int_{CV(t)} dV)}{dt} = \frac{dV}{dt} = 4\pi R^2 \frac{dR}{dt} \tag{4.12b}$$

  Over the CS, $(\mathbf{v} - \mathbf{v}_c).\mathbf{n} = 0$ except at station 2 where $(\mathbf{v} - \mathbf{v}_c).\mathbf{n} = +V_2$.
  We get

$$4\pi R^2 \frac{dR}{dt} + V_2 A = 0 \tag{4.12c}$$

- Bernoulli's equation
  Since the flow is inviscid, incompressible and steady, we can use steady Bernoulli across a streamline going from point 1 to point 2.

$$p_1 = p_a + \frac{1}{2}\rho V_2^2 \tag{4.12d}$$

- The pressure inside the bubble is given by Laplace's law

$$p_1 - p_a = \frac{4\sigma}{R(t)} \tag{4.12e}$$

  This equation results from the fact that for a very thin soap bubble in air, there are two air-soap interfaces: one on the inside of the bubble and the other on the outside, each having the same radius of curvature, R. Hence the total Laplace pressure within the bubble is twice what it would be for a bubble having a single interface (*e.g.* air bubble in water), *i.e.* $2 \times 2\sigma/R$. Combining Eq. 4.12d and Eq. 4.12e, we get

$$V_2 = 2\sqrt{\frac{2\sigma}{\rho R}} \tag{4.12f}$$

Inserting Eq. 4.12f into Eq. 4.12c yields

$$2\pi R^{\frac{5}{2}}\frac{dR}{dt} + \sqrt{\frac{2\sigma}{\rho}}A = 0 \tag{4.12g}$$

$$\frac{4\pi}{7}\frac{d(R^{\frac{7}{2}})}{dt} + \sqrt{\frac{2\sigma}{\rho}}A = 0 \tag{4.12h}$$

Integrating Eq. 4.12h with the initial condition $R(t=0) = R_0$ gives

$$\boxed{R(t) = (R_0^{\frac{7}{2}} - \frac{7}{4\pi}\sqrt{\frac{2\sigma}{\rho}}At)^{\frac{2}{7}}} \tag{4.12i}$$

(b) For the unsteady term in Euler's equation (or Bernoulli) to be negligible, we need

$$\frac{\partial V}{\partial t} \ll V\frac{\partial V}{\partial x}. \tag{4.12j}$$

Let l be a characteristic length scale in the x direction, τ a characteristic time scale and $\sqrt{\frac{\sigma}{\rho R_0}}$ a characteristic velocity.

From Eq. 4.12i, we see that a characteristic time scale for this process is $\tau = \frac{R_0^3}{A}\sqrt{\frac{\rho R_0}{\sigma}}$.
For the unsteady term to be negligible, we need

$$\tau \gg \frac{l}{\sqrt{\frac{\sigma}{\rho R_0}}} \tag{4.12k}$$

Which gives

$$\boxed{lA \ll R_0^3} \tag{4.12l}$$

The volume of the pipe needs to be negligible compared to the volume of the bubble for this process to be considered pseudo-steady.

□

Problem Solution by AH, Fall 2010

