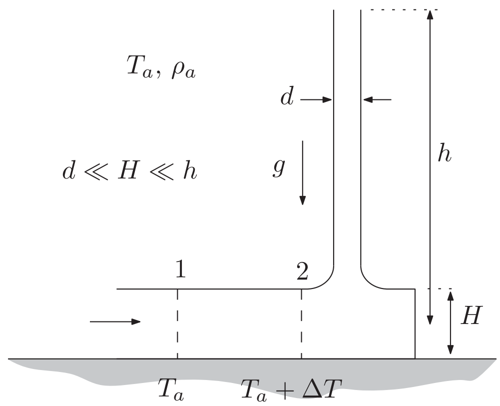
MIT Department of Mechanical Engineering

2.25 Advanced Fluid Mechanics

Problem 4.09

This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin



Consider a furnace of height H with a tall cylindrical smoke stack of diameter d ($d \ll H$) and height h ($h \gg H$). Air, an ideal gas ($P = \rho RT$), enters the furnace at atmospheric density and temperature and at local atmospheric pressure. Between stations 1 and 2, heat is added at constant pressure and the air temperature is raised by an amount ΔT. Thereafter, heat addition is negligible and the air rises through the stack at a sensibly constant density.

(a) On the assumption that viscous effects are negligible, derive an expression for the steady mass flow rate of air drawn by a stack of given height, h, in terms of the temperature rise in the furnace.

(b) If the chimney were capped off at the top, what would be the pressure differntial across the cap, assuming that ΔT would not be altered by the flow stoppage?

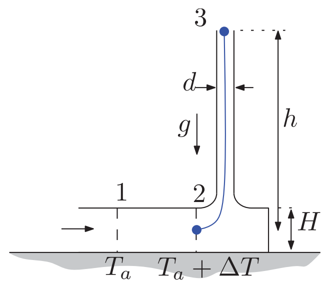
Note: The height h of the stack is small compared with the length RT_a/g over which the atmosphere density falls by $1/e$ (see Problem 1.8). Hence, gravitational density changes can be neglected.

Solution:

(a) First consider the effect of heat addition on the air density. Since the pressure at stations 1 and 2 is P_a, the heat is added at constant pressure such that

$$P_a = \rho_1 R T_1 = \rho_2 R T_2$$
$$\rho_a T_a = \rho_2 (T_a + \Delta T)$$
$$\Rightarrow \rho_2 = \frac{\rho_a}{1 + \Delta T / T_a}$$

Consider a streamline from station 2 to station 3:



As stated, the density is reasonably constant (*i.e.* $\rho_2 = \rho_3$) and viscous effects are negligible so we can apply Bernoulli's equation along the streamline shown above.

$$P_2 + \frac{1}{2}\rho_2 {v_2}^2 + \rho_2 g h_2 = P_3 + \frac{1}{2}\rho_2 {v_3}^2 + \rho_2 g h_3$$

If we assume that the streamlines at the exit of the smoke stack at station 3 are parallel, then we also set P_3 equal to the local ambient pressure at the top of the stack. Provided the air outside of the furnace and stack is isothermal at T_a and has roughly constant density ρ_a, then we can relate the pressure at station 3, P_3, to the ambient pressure, P_a, at ground level by stations 1 and 2 using our knowledge of hydrostatic pressure, such that $P_a = P_2 = P_3 + \rho_a g h$. Hence the pressure at the top of the smoke stack, P_3, is below the pressure at ground level, $P_a = P_1 = P_2$. Accordingly,

$$\frac{1}{2}\rho_2({v_3}^2 - {v_2}^2) = (P_2 - P_3) + \rho_2 g (h_2 - h_3)$$
$${v_3}^2 - {v_2}^2 = \frac{2}{\rho_2}(\cancel{P_3} + \rho_a g h - \cancel{P_3}) - 2gh$$
$${v_3}^2 - {v_2}^2 = 2\left(\frac{\rho_a}{\rho_2} - 1\right) gh$$

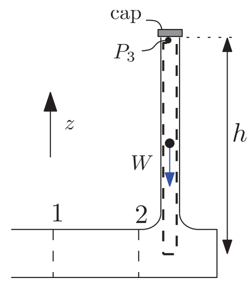
Conservation of mass from stations 2 to 3 tells us that

$$v_2 \sim \left(\frac{d}{H}\right)^2 v_3$$

Given that ($d \ll H$), we can neglect v_2 such that

$$v_3 = \sqrt{2\left(\frac{\rho_a}{\rho_2} - 1\right) gh} \qquad \Rightarrow \boxed{v_3 = \sqrt{2gh\frac{\Delta T}{T_a}}}$$

(b) If the chimney were capped, there would be no flow and we can apply our knowledge of fluid statics.



Consider a control volume around the air in the chimney. Note that this control volume is just below the cap such that the pressure at the top of the CV is not the local atmospheric pressure, but an unkown pressure P_3.

Static equilibrium gives:

$$\sum F_z = -W_{CV} + P_2(\pi d^2/4) - P_3(\pi d^2/4) = 0$$
$$-\rho_2 gh\cancel{(\pi d^2/4)} + P_a\cancel{(\pi d^2/4)} - P_3\cancel{(\pi d^2/4)} = 0$$
$$-\frac{\rho_a}{1+\Delta T/T_a} gh + P_a - P_3 = 0$$
$$P_3 = P_a - \frac{\rho_a}{1+\Delta T/T_a} gh$$

The ambient pressure above the cap is calculated from our knowledge of hydrostatic pressure as before, $P_{a,cap} = P_a - \rho_a gh$. Hence the pressure differential $\Delta P_{cap} = P_3 - P_{a,cap}$ across the cap is

$$\boxed{\Delta P_{cap} = \frac{\rho_a gh\Delta T}{T_a + \Delta T}}$$

□

Problem Solution by Tony Yu (2006), updated by Thomas Ober, Fall 2010

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