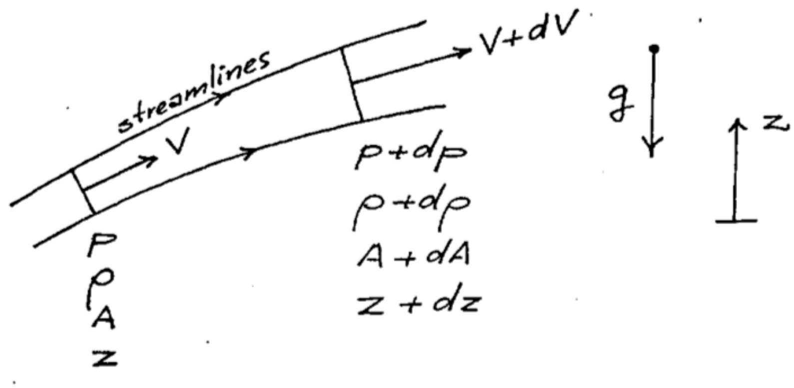
MIT Department of Mechanical Engineering

2.25 Advanced Fluid Mechanics

Problem 4.05

This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin



Consider the frictionless, steady flow of a compressible fluid in an infinitesimal stream tube.

(a) Demonstrate by the continuity and momentum theorems that

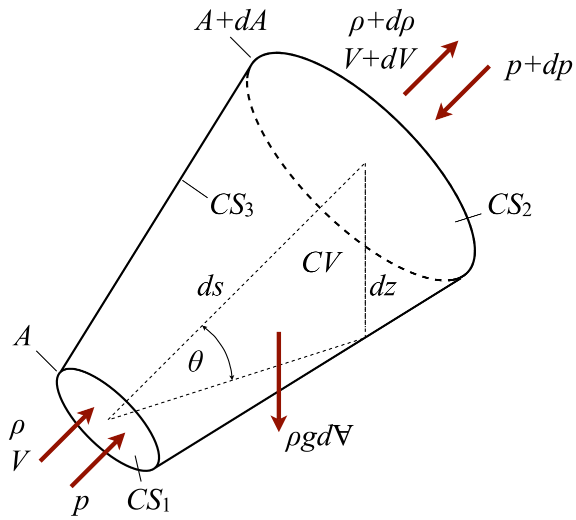
$$\frac{d\rho}{\rho}+\frac{dA}{A}+\frac{dV}{V}=0$$

$$dp+\rho V dV+\rho g dz=0$$

(b) Determine the integrated forms of these equations for an incompressible fluid.

(c) Derive the appropriate equations for unsteady frictionless, compressible flow, in a stream tube of cross-sectional area which depends on both space and time.

Solution:



(a) Here we consider an arbitrary control volume, CV, sitting along a streamline of length ds. For steady flow, we may write the integral mass conservation equation as

$$\int_{CS} \rho \mathbf{u} \cdot \hat{n} dA = 0 \tag{4.05a}$$

To evaluate this integral we must decompose it into three integrals for the three sub-control surfaces of this volume. For CS_1 located at the upstream portion of the CV, the integral is

$$\int_{CS_1} \rho \mathbf{u} \cdot \hat{n} dA = -\rho V A \tag{4.05b}$$

For CS_2 the result is

$$\int_{CS_2} \rho \mathbf{u} \cdot \hat{n} dA = (\rho + d\rho)(V + dV)(A + dA) = \rho V A + \rho V dA + \rho A dV + \cancel{\rho dV dA} + V A d\rho + \cancel{V d\rho dA} + \cancel{A d\rho dV} + \cancel{d\rho dV dA} \tag{4.05c}$$

where we have neglected higher order terms. There is no flow across CS_3 so

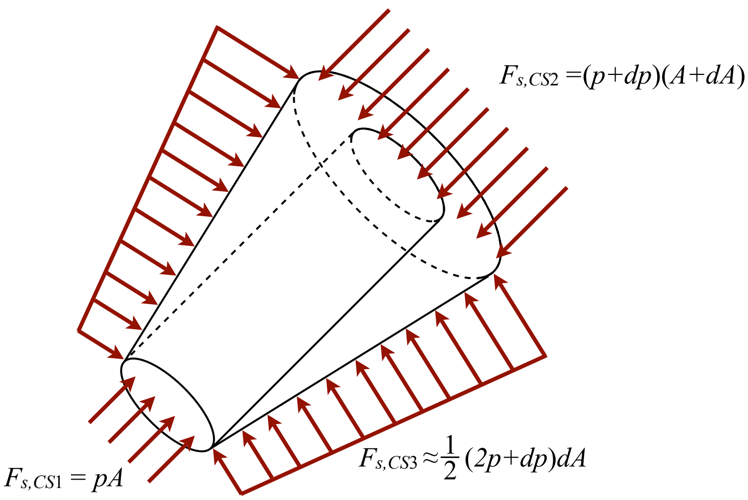
$$\int_{CS_3} \rho \mathbf{u} \cdot \hat{n} dA = 0 \tag{4.05d}$$

Combining Eq. (4.05b), (4.05c) and (4.05d) into Eq. (4.05a) we obtain

$$-\rho VA + \rho VA + \rho V dA + \rho A dV + VA d\rho = 0$$

Dividing this result by ρVA, we have

$$\boxed{\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0} \tag{4.05e}$$



For steady flow, the integral momentum conservation equation is

$$\int_{CS} \rho \mathbf{u} \Big(\mathbf{u} \cdot \hat{n}\Big) dA = \sum \mathbf{F} \tag{4.05f}$$

To calculate the left hand side of Eq. (4.05f), we calculate the momentum flux across CS_1

$$\int_{CS_1} \rho \mathbf{u} \Big(\mathbf{u} \cdot \hat{n}\Big) dA = -\rho V^2 A \tag{4.05g}$$

For CS_2 the result is

$$\int_{CS_2} \rho \mathbf{u} \Big(\mathbf{u} \cdot \hat{n}\Big) dA = (\rho + d\rho)(V + dV)^2(A + dA) \approx \rho V^2 A + 2\rho VA dV + V^2 A d\rho + \rho V^2 dA \tag{4.05h}$$

when we neglect higher order terms. There is no momentum flux across CS_3.

Now we must calculate the sum of the forces acting along the streamline direction. Since the flow is frictionless, the streamwise forces come only from pressure and gravity, hence

$$\sum \mathbf{F} \cdot \hat{s} = F_{gravity,s} + F_{pressure,s}$$

The gravitational force is

$$F_{gravity,s} = -\langle \rho \rangle d\forall g \sin\theta$$

where the angled brackets indicate the average value. Setting $\langle \rho \rangle = \frac{1}{2}\big(\rho+(\rho+d\rho)\big)$ and $\forall = \frac{1}{2}\big(A+(A+dA)\big)ds$ and $\sin\theta = \frac{dz}{ds}$, we obtain

$$F_{gravity,s} = -\frac{1}{4}\big(2\rho + d\rho\big)\big(2A + dA\big)gdz = -\rho A g dz - \cancel{\frac{1}{2}(\rho dA + d\rho A)gdz} - \cancel{\frac{1}{4}d\rho dA g dz} \tag{4.05i}$$

where we neglect all terms higher than first order. The force arising from the pressure acting on the control volume is

$$F_{pressure,s} = pA - (p + dp)(A + dA) + \langle p \rangle A_{CS_3} \sin\theta$$

where we set $\langle p \rangle = \frac{1}{2}\big(p + (p + dp)\big)$ and $A_{CS_3} \sin\theta = dA$. Having made these substitutions into the above equation we have

$$F_{pressure,s} = pA - (p + dp)(A + dA) + \frac{1}{2}\big(2p + dp\big)dA = -dpA - \cancel{\frac{1}{2}dpdA} \tag{4.05j}$$

where again we neglect the higher order term.

Combining Eq. (4.05g), (4.05h), (4.05i) and (4.05j) into Eq. (4.05f) we obtain

$$-\rho V^2 A + \rho V^2 A + 2\rho V A dV + V^2 A d\rho + \rho V^2 dA = -\rho A g dz - dpA$$

Eliminating terms and rearranging this result, we have

$$\rho A V dV + \rho V^2 A \left(\frac{dV}{V} + \frac{d\rho}{\rho} + \frac{dA}{A}\right) = -\rho A g dz - dpA \tag{4.05k}$$

Substituting Eq. (4.05e) into this result yields

$$\rho A V dV = -\rho A g dz - dpA \tag{4.05l}$$

Diving by A and rearranging we obtain

$$\boxed{dp + \rho V dV + \rho g dz = 0} \tag{4.05m}$$

(b) When we integrate Eq. (4.05e) from station 1 to station 2 on the streamline, we have

$$\int_{\rho_1}^{\rho_2} \frac{d\rho}{\rho} + \int_{A_1}^{A_2} \frac{dA}{A} + \int_{V_1}^{V_2} \frac{dV}{V} = 0 \tag{4.05n}$$

These integrals give

$$\ln\left(\frac{\rho_2}{\rho_1}\right) + \ln\left(\frac{A_2}{A_1}\right) + \ln\left(\frac{V_2}{V_1}\right) = \ln\left(\frac{\rho_2 V_2 A_2}{\rho_1 A_1 V_1}\right) = 0 \tag{4.05o}$$

This result may be rearranged to show

$$\boxed{\rho_1 A_1 V_1 = \rho_2 V_2 A_2} \tag{4.05p}$$

Again, when we integrate Eq. (4.05m) from station 1 to station 2 on the streamline, we have

$$\int_{p_1}^{p_2} dp + \int_{V_1}^{V_2} \rho V dV + \int_{z_1}^{z_2} \rho g dz = 0 \tag{4.05q}$$

Which gives the familiar Bernoulli equation

$$\boxed{p_2 - p_1 + \frac{1}{2}\rho\left(V_2^2 - V_1^2\right) + \rho g(z_2 - z_1) = 0} \tag{4.05r}$$

(c) For unsteady, frictionless, compressible flow, the integral mass conservation equation is

$$\int_{CV} \frac{\partial \rho}{\partial t} d\forall + \int_{CS} \rho \mathbf{u} \cdot \hat{n} dA = 0 \tag{4.05s}$$

The surface integrals in Eq. (4.05b), (4.05c) and (4.05d) remain valid, and the time varying volume integral is

$$\int_{CV} \frac{\partial \rho}{\partial t} d\forall = \frac{\partial \rho}{\partial t} A ds \tag{4.05t}$$

since in the limit $ds \to 0$, $dA \to 0$ and thus volume can be written as Ads. Combining Eq. (4.05b), (4.05c), (4.05d) and (4.05t) into Eq. (4.05s) and dividing by ρAV we obtain

$$\boxed{\frac{1}{\rho V}\frac{\partial \rho}{\partial t} ds + \frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0} \tag{4.05u}$$

For unsteady flow, the integral momentum conservation equation is

$$\int_{CV} \frac{\partial \rho \mathbf{u}}{\partial t}\, d\forall + \int_{CS} \rho \mathbf{u} \Big(\mathbf{u} \cdot \hat{n} \Big) dA = \sum \mathbf{F} \tag{4.05v}$$

The surface integrals and forces in Eq. (4.05g), (4.05h), (4.05i) and (4.05j) remain valid and the time dependent integral term is

$$\int_{CV} \frac{\partial \rho \mathbf{u}}{\partial t} d\forall = \frac{d}{dt} \Big(\rho V \Big) A ds \tag{4.05w}$$

again, since in the limit $ds \to 0$, $dA \to 0$ and thus volume is written as Ads. Combining Eq. (4.05g), (4.05h), (4.05i), (4.05j) and (4.05w) into Eq. (4.05v) we obtain

$$\frac{\partial}{\partial t} \Big(\rho V \Big) A ds - \rho V^2 A + \rho V^2 A + 2\rho V A dV + V^2 A d\rho + \rho V^2 dA = -\rho A g dz - dpA$$

Eliminating terms, expanding the time derivative, dividing by A, and rearranging the result, we have

$$\rho \frac{\partial V}{\partial t} ds + \rho V dV + \rho V^2 \left(\frac{1}{\rho V} \frac{\partial \rho}{\partial t} ds + \frac{dV}{V} + \frac{d\rho}{\rho} + \frac{dA}{A} \right) = -\rho g dz - dp$$

Substituting Eq. (4.05u) into the result above , rearranging and dividing by ρ we have

$$\boxed{\frac{\partial V}{\partial t} ds + \frac{dp}{\rho} + V dV + g dz = 0} \tag{4.05x}$$

Integrating Eq. (4.05x) from station 1 to station 2 on the streamline, we obtain the unsteady Bernoulli equation

$$\boxed{\int_{s_1}^{s_2} \frac{\partial V}{\partial t} ds + \int_{s_1}^{s_2} \frac{dp}{\rho} ds + \frac{1}{2} \Big(V_2^2 - V_1^2 \Big) + g(z_2 - z_1) = 0} \tag{4.05y}$$

If the fluid is incompressible, Eq. (4.05y) can be simplified into

$$\int_{s_1}^{s_2} \rho \frac{\partial V}{\partial t} ds + p_2 - p_1 + \frac{1}{2} \rho \Big(V_2^2 - V_1^2 \Big) + \rho g (z_2 - z_1) = 0 \tag{4.05z}$$

□

Problem Solution by Thomas Ober (2010), updated by Shabnam Raayai, Fall 2013

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