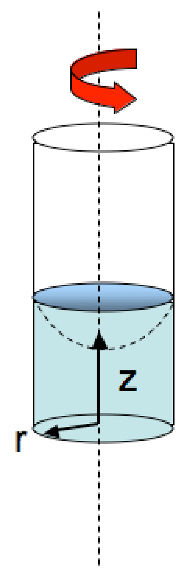
MIT Department of Mechanical Engineering

2.25 Advanced Fluid Mechanics

Problem 1.14

This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin

Cylinder with Liquid Rotating



- (a) Demonstrate that when a cylindrical can of liquid rotates like a solid body about its vertical axis with uniform angular velocity, ω, the free surface is a parabolic of revolution.
- (b) Demonstrate that the pressure difference between any two points in the fluid is given by

$$p_2 - p_1 = \rho g(z_2 - z_1) + \rho\omega^2(r_2^2 - r_1^2)/2, \qquad (1.14a)$$

  where z is elevation and r is the radial distance from the axis.
- (c) How would the results differ if the can were of square cross section?

Solution:

- (a) If the fluid rotates like a solid body, then

$$a_r = \frac{V_\theta^2}{r} = \omega^2 r, \tag{1.14b}$$

then, for the fluid

$$\frac{\partial p}{\partial r} = \rho\omega^2 r, \tag{1.14c}$$

and now, considering gravity,

$$\frac{\partial p}{\partial z} = -\rho g. \tag{1.14d}$$

At the surface,$\Delta\ p = 0$, then $\frac{\partial p_s}{\partial r_s}\delta r_s + \frac{\partial p_s}{\partial z_s}\delta z_s = 0$, or $\rho\omega^2 r_s \delta r_s - \rho g \delta z_s = 0$, then

$$\omega^2 r_s \delta r_s = g\delta z_s, \Rightarrow \int \omega^2 r_s \delta r_s = \int g\delta z_s, \Rightarrow z_s = \frac{\omega^2 r_s^2}{2g} + Const, \tag{1.14e}$$

then the surface is a revolution paraboloid.

- (b) Now, let's integrate the radial derivative and differentiate with respect to the axial coordinate to compare the equations,

$$\frac{\partial p}{\partial r} = \rho\omega^2 r, \Rightarrow p(r,z) = \rho\omega^2\frac{r^2}{2} + f(z), \Rightarrow \frac{\partial p}{\partial z} = \frac{\partial f}{\partial z}, \tag{1.14f}$$

Now, comparing both expressions for $\frac{\partial p}{\partial z}$, we notice that $\rho g = \frac{\partial f}{\partial z}$, then $f = -\rho g z + Const$. Finally,

$$p(z,r) = \rho\omega^2\frac{r^2}{2} - \rho g z + Const, \tag{1.14g}$$

then, for two different points inside the liquid,

$$p(z_2,r_2) - p(z_1,r_1) = \rho\omega^2\left(\frac{r_2^2}{2} - \frac{r_1^2}{2}\right) - \rho g(z_2 - z_1), \tag{1.14h}$$

- (c) No practical difference, the surface just would be cut by two planes instead of a cylinder (the effects of surface tension would be different, but this is unimportant as long as the surface tension contribution is small).

□

Problem Solution by MC, Fall 2008

