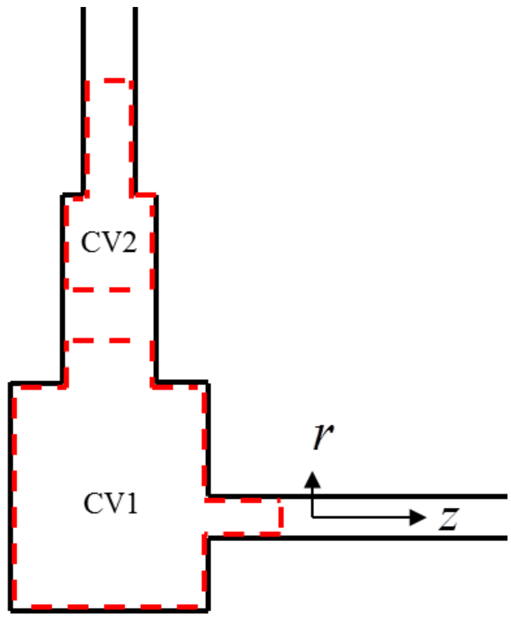
Part (a) (2 points)



Consider the two control volumes shown above. Steady mass conservation in CV1 gives:

$$\iint_{capillary} \rho \mathbf{v}\cdot\mathbf{n}dA + \iint_{needle} \rho \mathbf{v}\cdot\mathbf{n}dA = 0 \tag{1.1}$$

$$\Rightarrow \left(v_{eo} - \bar{v}_b\right)\frac{\pi D_c^2}{4} = \bar{v}_n \frac{\pi D_n^2}{4}. \tag{1.2}$$

Similarly, in CV2:

$$\iint_{needle} \rho \mathbf{v}\cdot\mathbf{n}dA + \iint_{\mu syringe} \rho \mathbf{v}\cdot\mathbf{n}dA = 0 \tag{1.3}$$

$$\Rightarrow \bar{v}_n \frac{\pi D_n^2}{4} = \bar{v}_m \frac{\pi D_m^2}{4}. \tag{1.4}$$

Note that all of the storage terms $\left(d/dt\iiint \rho dV\right)$ vanish because the flow is steady. Setting (1.2) and (1.4) equal and simplifying,

$$\boxed{\left(v_{eo} - \bar{v}_b\right)D_c^2 = \bar{v}_n D_n^2 = \bar{v}_m D_m^2.} \tag{1.5}$$

Part (b) (3 points)

Assumptions

- The problem statement provided the assumptions of steady, fully developed, and incompressible flow in the capillary tube. Thus all $\partial / \partial t = 0$ and $\partial / \partial z = 0$, except $\partial p / \partial z \neq 0$.
- Assume the flow is *axisymmetric*, so that all $\partial / \partial \theta = 0$.
- Assume the flow is *unidirectional* (no r or θ component), and postulate a solution of the form

$$\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ v_z(r) \end{bmatrix}.$$

Following the figure in part (a), let us use cylindrical coordinates with z increasing from left to right. The Navier-Stokes equation (z component) is

$$\rho\left(\underbrace{\frac{\partial v_z}{\partial t}}_{=0,\text{ steady}} + \underbrace{v_r \frac{\partial v_z}{\partial r}}_{=0, v_r=0} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}_{=0, v_\theta=0, \partial/\partial\theta=0} + \underbrace{v_z \frac{\partial v_z}{\partial z}}_{=0,\text{ fully developed}}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}_{=0, \partial/\partial\theta=0} + \underbrace{\frac{\partial^2 v_z}{\partial z^2}}_{=0,\text{ fully developed}}\right] + \underbrace{\rho g_z}_{=0, g_z=0} \quad (1.6)$$

Under the assumptions we've made, the equation becomes

$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{\partial p}{\partial z} = \text{Constant.} \quad (1.7)$$

Boundary conditions

1. Centerline ($r = 0$): either $dv_z / dr = 0$ or v_z is finite.
2. Wall ($r = D_c / 2 \equiv R_c$): $v_z = -v_{eo}$. (electro-osmotic slip)

The general solution is obtained by rearranging (1.7) and integrating twice:

$$v_z = \frac{1}{4\mu}\frac{\partial p}{\partial z} r^2 + c_1 \ln r + c_2. \quad (1.8)$$

The first BC implies that $c_1 = 0$. The second BC implies that

$$c_2 = -\frac{1}{4\mu}\frac{\partial p}{\partial z} R_c^2 - v_{eo}.$$

Thus the solution is

$$\boxed{v_z(r) = \frac{R_c^2}{4\mu}\left(-\frac{\partial p}{\partial z}\right)\left(1-\frac{r^2}{R_c^2}\right) - v_{eo}.} \tag{1.9}$$

Equation (1.9) is a linear superposition of the solutions for Poiseuille flow and the spatially uniform "plug flow" due to electro-osmosis.

Note: There was an error in the wording of this part. It asked for the velocity profile in terms of the "pressure gradient *and* $\bar{v}_b$." It should have said "pressure gradient *or* $\bar{v}_b$." As long as your answer was equivalent to (1.9) or (1.11), it was given credit, regardless of the choice of variables.

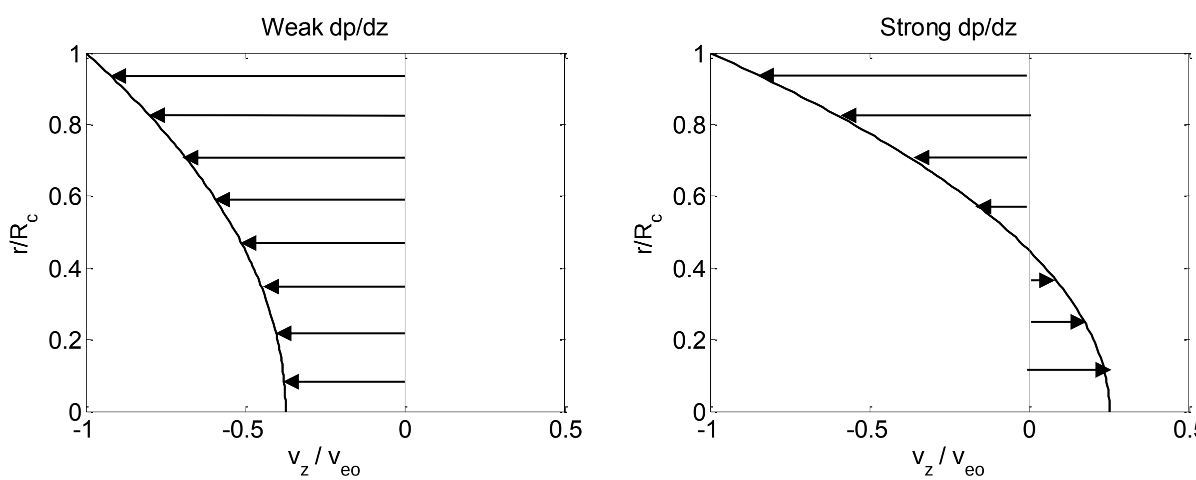
As discussed in class, the average velocity in Poiseuille flow is equal to half of the maximum value, i.e. the value at the centerline. That is,

$$\bar{v}_b = \frac{v_{b,\max}}{2} = -\frac{R_c^2}{8\mu}\frac{\partial p}{\partial z} = -\frac{D_c^2}{32\mu}\frac{\partial p}{\partial z}. \tag{1.10}$$

Thus the velocity profile could be expressed in terms of $\bar{v}_b$ as

$$\boxed{v_z(r) = 2\bar{v}_b\left(1-\frac{r^2}{R_c^2}\right) - v_{eo}.} \tag{1.11}$$

Your sketches needed to show a discernible difference in the relative dominance of the electro-osmotic and pressure-driven components. One possible solution is shown below:



Note also that the coordinate system used here was not the only choice. For example, we could let *z* increase from right to left. In this case, the solution will be exactly the same as in (1.9) or (1.11) except the sign of every term on the right-hand side will be flipped.

Part (c) (2 points)

Part (b) showed that there are two contributions to the volume flow rate in the capillary: electro-osmotic flow and the pressure-driven backflow. The volume flow rate associated with the backflow is given by the Hagen-Poiseuille equation:

$$Q_{backflow} = \frac{\pi D_c^4}{128\mu}\left(\frac{\Delta p}{\Delta z}\right) = \frac{\pi D_c^4 \left(p_1 - p_{atm}\right)}{128\mu L_c}. \tag{1.12}$$

This volume flow rate is related to the average backflow velocity $\overline{v}_b$ by

$$\frac{\pi D_c^4 \left(p_1 - p_{atm}\right)}{128\mu L_c} = \overline{v}_b \frac{\pi D_c^2}{4}. \tag{1.13}$$

Thus the pressure difference along the capillary is

$$p_1 - p_{atm} = \frac{32\overline{v}_b \mu L_c}{D_c^2}. \tag{1.14}$$

We know from the problem statement that the pressure drop across the needle and microsyringe together is also $\left(p_1 - p_{atm}\right)$. Since the individual pressure drops across the needle and microsyringe are additive, we can also write

$$p_1 - p_{atm} = \frac{32\overline{v}_m \mu L_m}{D_m^2} + \frac{32\overline{v}_n \mu L_n}{D_n^2}. \tag{1.15}$$

Setting (1.14) and (1.15) equal and simplifying,

$$\boxed{\frac{\overline{v}_b L_c}{D_c^2} = \frac{\overline{v}_m L_m}{D_m^2} + \frac{\overline{v}_n L_n}{D_n^2}.} \tag{1.16}$$

Part (d) (1 point)

In this part, we are effectively replacing the needle+microsyringe with a new microsyringe with the same pressure drop, diameter, and average velocity, which will require it to have a different length, L_{eff}. Thus we can use the same strategy of part (c) and apply the Hagen-Poiseuille equation across this new microsyringe. We want to find L_{eff} such that

$$\frac{\overline{v}_m L_m}{D_m^2} + \frac{\overline{v}_n L_n}{D_n^2} = \frac{\overline{v}_m L_{eff}}{D_m^2}. \tag{1.17}$$

Noting from part (a) (equation (1.5)) that $\overline{v}_n = \overline{v}_m \left(D_m / D_n\right)^2$, we can rewrite this expression as

$$\frac{\overline{v}_m L_m}{D_m^2} + \frac{\overline{v}_m L_n}{D_n^4} D_m^2 = \frac{\overline{v}_m L_{eff}}{D_m^2} \tag{1.18}$$

Thus (1.17) is satisfied if

$$L_{eff} = L_m + L_n \left(\frac{D_m}{D_n} \right)^4 = L_m \left[1 + \frac{L_n}{L_m} \left(\frac{D_m}{D_n} \right)^4 \right]. \tag{1.19}$$

Part (e) (3 points)

We seek two unknowns: $\overline{v}_m$ and $\overline{v}_b$. These quantities are related to one another by the mass balance (1.5)

$$\overline{v}_m D_m^2 = \left(v_{eo} - \overline{v}_b \right) D_c^2. \tag{1.20}$$

Another relationship is given by setting (1.16) and (1.17) equal:

$$\frac{\overline{v}_b L_c}{D_c^2} = \frac{\overline{v}_m L_{eff}}{D_m^2}. \tag{1.21}$$

(1.20) and (1.21) are two equations with two unknowns, $\overline{v}_m$ and $\overline{v}_b$. Solving (1.21) for $\overline{v}_b$ and inserting into (1.20),

$$\overline{v}_m = v_{eo} \left(\frac{D_c}{D_m} \right)^2 - \overline{v}_b \left(\frac{D_c}{D_m} \right)^2 = v_{eo} \left(\frac{D_c}{D_m} \right)^2 - \overline{v}_m \frac{L_{eff}}{L_c} \left(\frac{D_c}{D_m} \right)^4. \tag{1.22}$$

Rearranging, we have

$$\overline{v}_m \left[1 + \frac{L_{eff}}{L_c} \left(\frac{D_c}{D_m} \right)^4 \right] = v_{eo} \left(\frac{D_c}{D_m} \right)^2 \tag{1.23}$$

or

$$\boxed{\frac{\overline{v}_m}{v_{eo}} = \frac{1}{\left(D_m / D_c \right)^2 + \left(L_{eff} / L_c \right) \left(D_c / D_m \right)^2}.} \tag{1.24}$$

Similarly,

$$\frac{\overline{v}_b}{v_{eo}} = \frac{L_{eff}}{L_c} \left(\frac{D_c}{D_m} \right)^2 \frac{\overline{v}_m}{v_{eo}} = \frac{L_{eff}}{L_c} \left(\frac{D_c}{D_m} \right)^2 \frac{1}{\left(D_m / D_c \right)^2 + \left(L_{eff} / L_c \right) \left(D_c / D_m \right)^2}. \tag{1.25}$$

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Simplifying,

$$\boxed{\frac{\bar{v}_b}{v_{eo}} = \frac{1}{1+\left(D_m / D_c\right)^4 \left(L_c / L_{eff}\right)}.} \tag{1.26}$$

Part (f) (2 points)

The pressure drop is maximized when the net volume flow rate through the capillary is zero:

$$Q_{net} = 0 = \frac{\pi D_c^4 \left(\Delta p\right)_{\max}}{128 \mu L_c} - v_{eo} \frac{\pi D_c^2}{4}. \tag{1.27}$$

Thus

$$\boxed{\left(\Delta p\right)_{\max} = \frac{32 \mu L_c v_{eo}}{D_c^2}.} \tag{1.28}$$

Note that a vanishing volume flow rate implies that $\bar{v}_b = v_{eo}$. Thus we can also obtain the correct answer by simply substituting $\bar{v}_b = v_{eo}$ into the rearranged Hagen-Poiseuille equation, (1.14).

Part (g) (2 points)

Setting $\bar{v}_m = dL_{eff} / dt$, we can write equation (3) in the problem statement as

$$\frac{dL_{eff}}{dt} = \frac{v_{eo}}{\left(D_m / D_c\right)^2 + \left(D_c / D_m\right)^2 \left(L_{eff} / L_c\right)}. \tag{1.29}$$

This ordinary differential equation can be solved analytically, or one can make a simple dimensional argument that $\boxed{L_{eff} / t \propto 1 / L_{eff} \Rightarrow L_{eff} \propto t^{1/2}.}$

To solve the ODE, let us define $\alpha = v_{eo}$, $\beta = \left(D_m / D_c\right)^2$, and $\gamma = \left(1 / L_c\right)\left(D_c / D_m\right)^2$ to clean up the algebra. Then (1.29) becomes

$$\frac{dL_{eff}}{dt} = \frac{\alpha}{\beta + \gamma L_{eff}}. \tag{1.30}$$

It can be readily verified that (1.30) has the solutions

$$L_{eff} = \frac{-\beta \pm \sqrt{\beta^2 + \gamma\left(2\alpha t + c_1\right)}}{\gamma}, \tag{1.31}$$

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where c_1 is a constant of integration that depends on the initial height of the fluid in the needle and syringe. Since L_{eff} is positive and increasing with time, the physical solution is the one with the positive coefficient of t. Converting back to the original constants, we have

$$L_{eff}(t) = L_c \frac{-(D_m / D_c)^2 + \sqrt{(D_m / D_c)^4 + 2(v_{eo} / L_c)(D_c / D_m)^2 t + c_1'}}{(D_c / D_m)^2} \tag{1.32}$$

where $c_1' = c_1 \gamma$. Thus

$$\boxed{n = \frac{1}{2}.}$$

Note that this result also implies that $L_m \propto t^{1/2}$ since

$$\frac{dL_{eff}}{dt} = \frac{dL_m}{dt}.$$

Thus (1.29) could also have been solved with L_m as the dependent variable rather than L_{eff}.

7

Solution to Problem 2-Quiz 2 2013

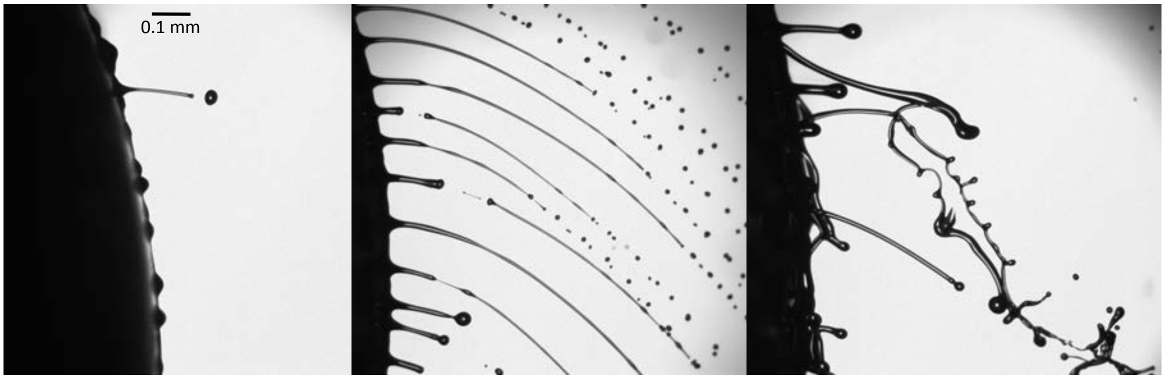

Figure 1: Rotary atomization at different flow rates (Image: Bavand-HML).

(a):
We know that the following independent properties are important for determining the value of h_R: radius of the cup (R), viscosity (μ), density (ρ), surface tension (σ), volumetric flow rate (Q), and angular speed of the cup (Ω)[1]. So, for a fixed given θ_0, the following holds:

$$h_R = fun(R, \mu, \rho, \sigma, Q, \Omega)$$

thus, $n = 7$ and knowing that there are three dimensions involved in these parameters ([M],[L], and [T] $\Rightarrow r = 3$) we can conclude that at a fixed value of θ_0 there are 4 dimensionless groups. We select R,ρ, and σ as the repeating parameters and using Buckingham-Pi theorem following dimensionless groups will be identified:

$$\boxed{\begin{aligned} &\Pi_1 = h_R/R \\ &\Pi_2 = \mu/\sqrt{\rho\sigma R} \\ &\Pi_3 = Q\sqrt{\rho/(\sigma R^3)} \\ &\Pi_4 = \Omega\sqrt{\rho R^3/\sigma} \end{aligned}}$$

(b):
Writing down the conservation of mass we will have:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 v_r) + \frac{1}{r sin\theta}\frac{\partial}{\partial \theta}(\rho v_\theta sin\theta) + \frac{1}{r sin\theta}\frac{\partial}{\partial \phi}(\rho u_\phi) = 0$$

we already know that due to axi-symmetry $\partial/\partial\phi = 0$ thus the continuity equation simplifies to:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 v_r) + \frac{1}{r sin\theta}\frac{\partial}{\partial \theta}(\rho v_\theta sin\theta) = 0$$

[1] the angle of the cup (θ_0) is another important parameter and we can keep it separate since it is already dimensionless

now knowing the scale of different parameters ($v_r \sim V_R, v_\theta \sim V_\theta, r \sim R, \theta \sim \delta\theta \sim h/R$), we can proceed with a scaling argument:

$$\frac{V_R}{R} \sim \frac{V_\theta}{R\delta\theta} \Rightarrow \boxed{V_\theta \sim \delta\theta V_R \sim \frac{h}{R} V_R \Rightarrow v_\theta << v_r}$$

(c):
Writing down N.S.E in spherical coordinates:

$$\frac{\partial vr}{\partial t} + (\underline{v}.\underline{\nabla}).v_r - \frac{v_\theta^2 + v_\phi^2}{r} = \frac{-1}{\rho}\frac{\partial P}{\partial r} + \nu\left[\nabla^2 v_r - \frac{2v_r}{r^2} - \frac{2}{r^2 sin\theta}\frac{\partial(v_\theta sin\theta)}{\partial\theta} - \frac{2}{r^2 sin\theta}\frac{\partial v_\phi}{\partial\phi}\right.$$

now using the appropriate scales for length and velocities involved ($v_r \sim V_R, v_\theta \sim V_\theta \sim \delta\theta V_R << V_R, r \sim R, \theta \sim \delta\theta << 1, t \sim \tau \sim \infty$), the Also using the problem's hint or N.S.E for the $\theta-$component it is easy to show that within the thin layer pressure is constant and atmospheric ($P \sim P_a$), thus N.S.E. for the $r-$component will be simplified to:

$$\frac{V_R}{\infty}O(1) + V_R\frac{V_R}{R}O(1) + V_R\frac{V_R}{R}O(1) + 0 - (0 + Rsin^2\theta_0\Omega^2) =$$
$$0 + \nu\ \frac{V_R}{R^2}O(1) + \frac{V_R}{R^2(\delta\theta)^2}O(1) + 0 - \frac{V_R}{R^2}O(1) - \frac{V_R}{R^2}O(1) - 0$$

If $\delta\theta << 1$ or $(r\delta\theta/r) << 1$ then the dominant viscous term will be:

$$\boxed{if\ (\delta\theta)^2 << 1 \Rightarrow Dominant\ Viscous\ Term:\ \frac{\nu}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial v_r}{\partial\theta}\right)} \tag{1}$$

For ignoring the inertia convection terms the criterion will be:

$$\boxed{if\ \frac{\rho V_R R\delta\theta}{\mu}(\delta\theta) << 1\ \ or\ equally\ if:\ \frac{\rho V_R h}{\mu}\frac{h}{R} = Re_h\frac{h}{R} << 1} \tag{2}$$

For ignoring the temporal derivative the criterion will be:

$$\boxed{if\ \tau \to \infty \Rightarrow \frac{R^2\delta\theta^2}{\nu\tau} = 0} \tag{3}$$

One may argue that based on the selection of the reference frame the time scale may change from ∞ to $1/\Omega$, in that case the condition in Equation (3) will change to:

$$\frac{R^2\delta\theta^2\Omega}{\nu} << 1$$

With the three mentioned criteria one can see that all the terms on the left hand side of the N.S.E. ($r-$component) other than the centripetal acceleration vanish, the scaling for centripetal acceleration shows that compared to the viscous terms they may be large enough and we have to keep them:

$$\frac{Centripetal}{Viscous} \sim \frac{Rsin\theta^2\Omega^2}{\nu V_R/R^2\delta\theta^2} \sim O(1)$$

after simplifying the N.S.E ($r-$component) we will be left with the following:

$$\frac{v_\phi^2}{r} + \frac{\nu}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial v_r}{\partial\theta}\right) = 0$$

since $\theta \sim \theta_0 = const.$ in the thin film, then we can further simplify the N.S.E into:

$$\boxed{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\rho \Omega^2 r sin^2\theta_0}{\mu} = 0} \tag{4}$$

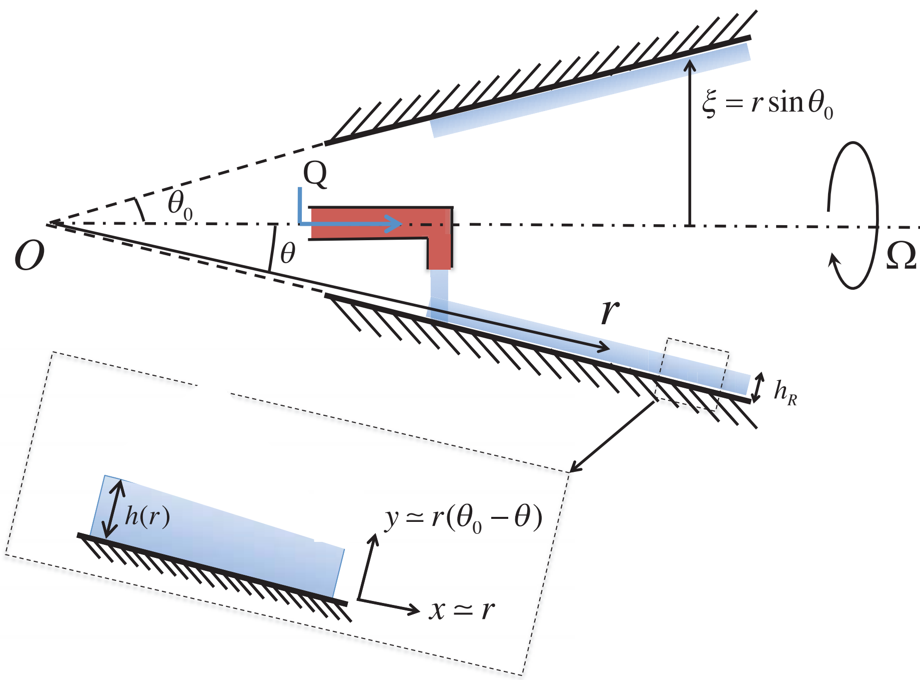

Figure 2: Schematics of the problem.

(d):
As shown in Figure 2 we move to a local cartesian coordinate system $(x - y)$ and the boundary conditions for $y = 0$will be:

$$v_\theta = 0$$
$$v_\phi = rsin\theta_0\Omega$$
$$v_r = 0$$

and for $y = h(r)$:

$$v_\theta = ?$$
$$v_\phi \ rsin\theta_0\Omega$$
$$\frac{\partial v_r}{\partial y} = \frac{1}{r}\frac{\partial v_r}{\partial \theta} v_r = 0$$

(e):

Equation (4) in the $x - y$ coordinate system will be:

$$\frac{\partial^2 v_x}{\partial y^2} + \frac{\rho\Omega^2 x sin^2\theta_0}{\mu} = 0$$

after integrating we will get:

$$\boxed{v_r = \frac{\rho\Omega^2 sin^2\theta_0 r}{\mu}\left(h(r)y - y^2/2\right)} \tag{5}$$

(f):
In order to find the volumetric flow rate we just need to integrate the velocity along the liquid height:

$$Q = 2\pi(rsin\theta_0)\int_0^{h(r)} v_r dy = 2\pi rsin\theta_0 \int_0^{h(r)} \frac{\rho\Omega^2 sin^2\theta_0 r}{\mu}(hy - y^2/2)dy$$

which can be simplified to:

$$Q(r) = (2\pi rsin\theta_0)\left(\frac{\rho\Omega^2 sin^2\theta_0 r}{\mu}\frac{h^3}{3}\right)$$

Thus $h(r)$ can be shown to be:

$$\boxed{h(r) = \left(\frac{3\mu Q}{2\pi\rho\Omega^2 r^2 sin^2\theta_0}\right)^{1/3}} \tag{6}$$

(g):
In order to find the pathline we use the definition of a pathline:

$$\frac{dr_s}{dt} = v_r$$
$$\frac{d\phi_s}{dt} = \Omega$$

which will lead to:

$$\frac{dr_s}{d\phi_s} = \frac{v_r}{\Omega}$$

For a particle or bubble at the liquid surface we have:

$$v_r(y = h(r)) = \frac{\rho\Omega^2 sin^2\theta_0 r}{\mu}\frac{h^2}{2} \tag{7}$$

and we also know that:

$$h(r) = \left(\frac{3\mu Q}{2\pi\rho\Omega^2 r^2 sin^2\theta_0}\right)^{1/3} \tag{8}$$

thus one can easily show that:

$$\frac{dr_s}{d\phi_s} = \left(\frac{9\rho Q^2}{32\pi^2\mu\Omega}\right)^{1/3} r_s^{-1/3}$$

which results in:

$$\boxed{\frac{dr_s}{d\phi_s} = A r_s^{-1/3} \text{ in which } A = \left(\frac{9\rho Q^2}{32\pi^2\mu\Omega}\right)^{1/3}}$$

Using this relationships one can actually follow the spirals that the liquid make in the bell atomizer and find an estimate for the spacing of the spirals by solving for the change in r_s over a 2π change in ϕ. The algebra will give:

$$\Delta r_s = \left(\frac{16\rho Q^2}{\mu\Omega}\right)^{1/4}$$

or in the dimensionless form:

$$\Delta r_s = \left(\frac{16\Pi_3^2}{\Pi_2\Pi_4}\right)^{1/4}$$

Figure 3 shows a comparison between the predictions from theory and the experimental measurements for the spiral's spacing.

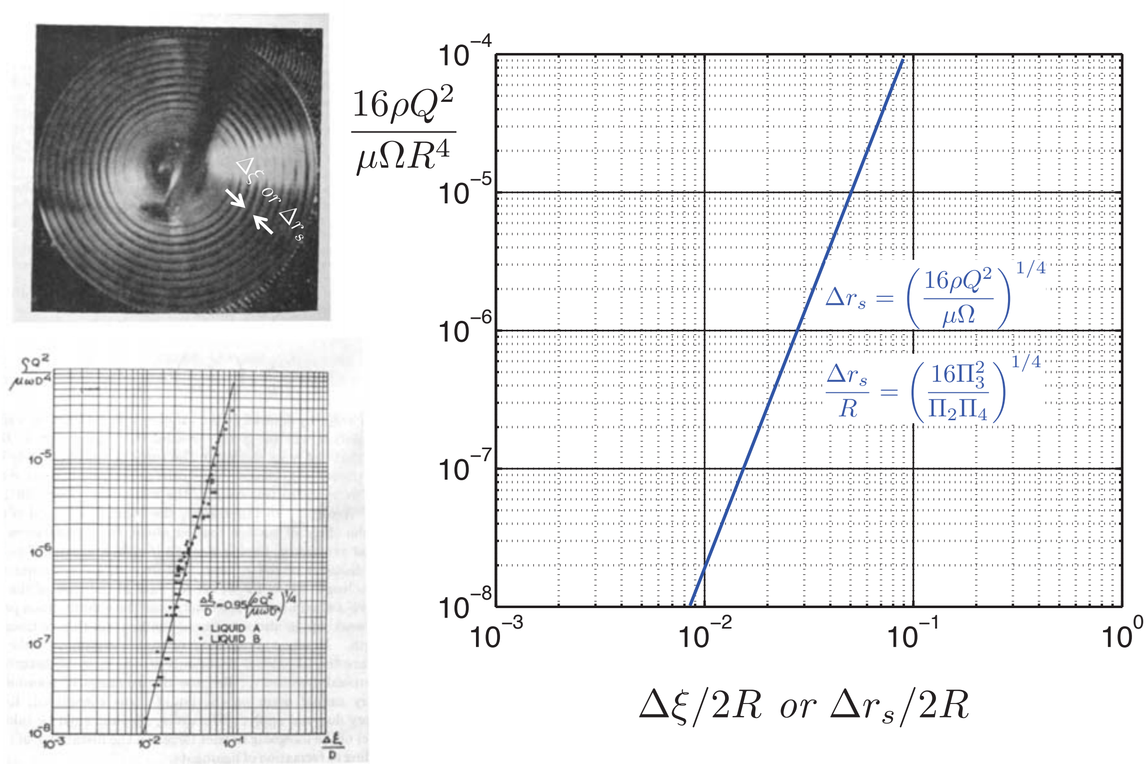

Figure 3: Results from Hinze et al. ([]) showing the agreement between measurements of the liquid's spiral spacing and the lubrication theory predictions.

Extra information:
One aspect of this problem which we did not cover is the behavior of liquid drops/ligaments/film at the rim of the cup. Experimentally the formation of drops, ligaments, and film at the rim has been seen at different working conditions (Figure 1 and 5(a,b,c)). It is interesting to observe that all the experimental data maps on a phase diagram which is made by combination of dimensionless numbers you got for this problem in part (a) (Figures 4 and 5).

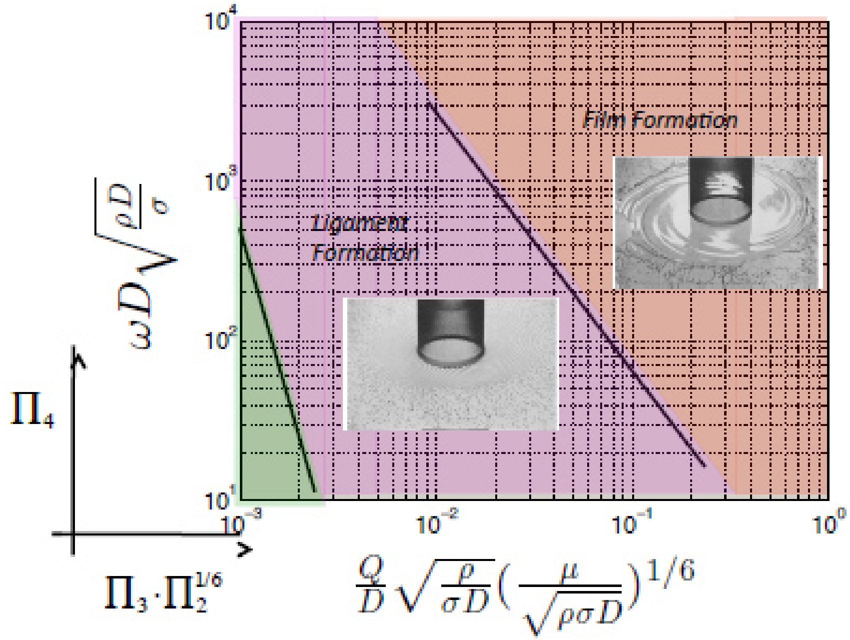

Figure 4: Results from Hinze et al. ([]) showing the performance map for the rotary atomization described with dimensionless groups introduced in this problem.

Table 1 summarizes all the important parameters acting on different control surfaces for the selected control volume:

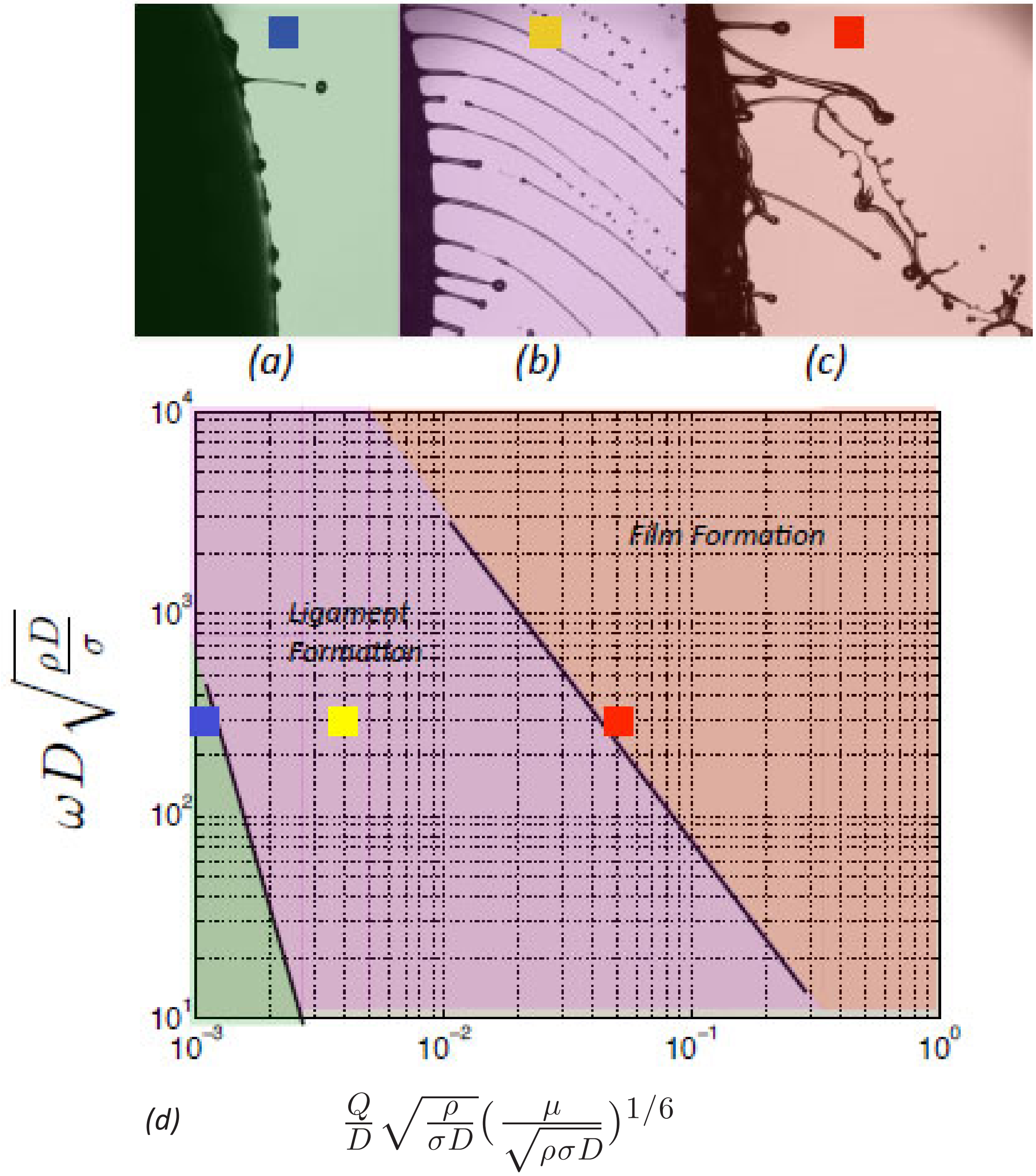

Figure 5: (a) Single drop formation stage (b) Ligament formation stage (c) Liquid film formation stage. (d) Results from Hinze et al. ([]) showing the performance map for the rotary atomization described with dimensionless groups introduced in this problem.