Solution to Problem 1-Final Exam- Fall 2013

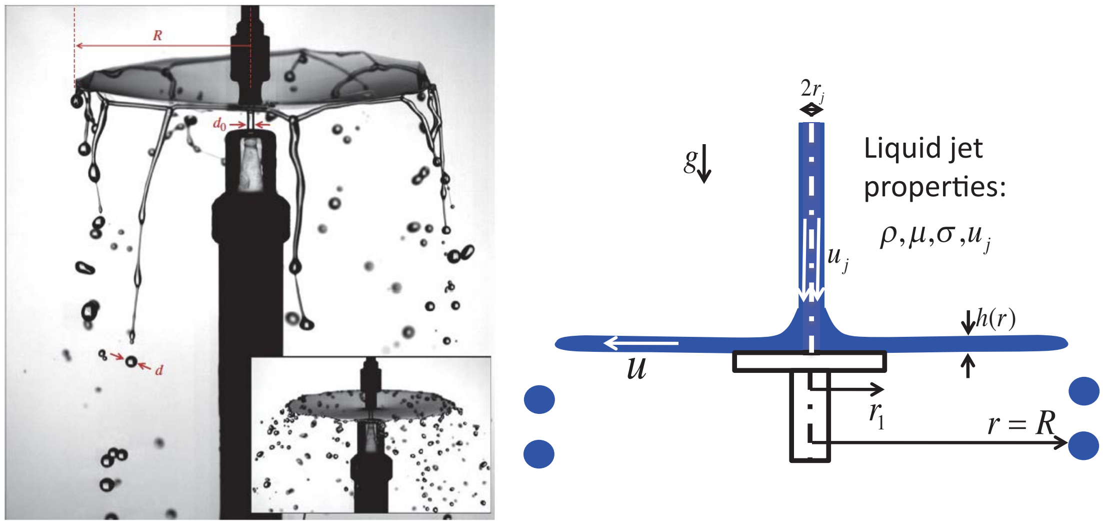

Figure 1: Viscous "Savart Sheet". Image courtesy: Villermaux et. al. [1]. This kind of geometry was first studied by Felix Savart in 1833.

(a):
We know that the following independent properties are important for determining the value of R: radius of the disk (r_1), radius of the jet (r_j), viscosity (μ), density (ρ), surface tension (σ), velocity of the jet (u_j), and gravity (g), so:

$$R = f(r_1, r_j, \mu, \rho, \sigma, u_j, g)$$

thus, $n = 8$ and knowing that there are three dimensions involved in these parameters ($[M]$,$[L]$, and $[T] \Rightarrow r = 3$) we can conclude that there are 5 dimensionless groups. We select r_1,v_j, and σ as the repeating parameters and using Buckingham-Pi theorem following dimensionless groups will be identified:

$$\boxed{\begin{aligned} \Pi_1 &= R/r_1 \\ \Pi_2 &= r_j/r_1 \\ \Pi_3 &= \mu u_j/\sigma \\ \Pi_4 &= g r_1/u_j^2 \\ \Pi_5 &= \rho u_j^2 r_1/\sigma \end{aligned}}$$

So the relationship for R can be written in the following dimensionless form:

$$R/r_1 = f\left(\frac{r_j}{r_1}, \frac{\mu u_j}{\sigma}, \frac{g r_1}{u_j^2}, \frac{\rho u_j^2 r_1}{\sigma}\right) \tag{1}$$

One can easily see that Π_3 is the Capillary number (comparing the jet velocity with visco-capillary velocity scale) and Π_4 is the inverse of Froude number (comparing the gravity and inertia). The other dimensionless number found in this problem, (Π_5) is indeed the Weber number comparing the inertia stresses with capillary pressure. Thus, in order to ignore the effects of gravity and viscosity we have to satisfy the following[1]:

$$\begin{aligned} &\textit{for ignoring gravity: } Fr^{-1} \equiv \frac{gr_1}{u_j^2} << 1 \\ &\textit{for ignoring viscosity: } Re \equiv \frac{Ca}{We} << 1 \end{aligned} \tag{2}$$

[2] Satisfying the two mentioned criteria, Equation 1 will be simplified to:

$$\boxed{R/r_1 = f\left(\frac{r_j}{r_1}, \frac{\rho u_j^2 r_1}{\sigma}\right)} \tag{3}$$

(b):
From conservation of mass written for the dashed control volume in Figure 2 we will have (note that since viscous losses are ignored in this part, by Bernoulli, we can say that velocity at the edge of the disk is equal to the jet velocity i.e. along a streamline the velocity is unchanged.):

$$\textit{Mass. Cons.: } \rho u_j^2(\pi r_j^2) = \rho u_j(2\pi r_1 h(r_1)) \rightarrow \boxed{h(r_1) = r_j^2/2r_1} \tag{4}$$

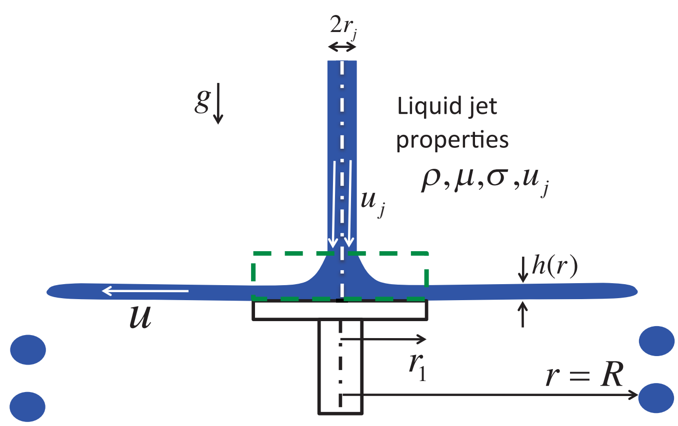

Figure 2: Simplified geometry with the selected control volume in dashed green line.

For calculating the thrust force on the disc we need to apply the conservation of linear momentum for y-component, lets assume that the disc is exerting Fe_y to the control volume:

$$\textit{Momentum. Cons.: } Fe_y = \rho v_j^2(\pi r_j^2)e_y$$

[1] Another approach to do the dimensional analysis is picking ρ, σ, r_1 as the repeating parameters and that will lead to $R/r_1 = g(r_j/r_1, Oh = \mu/\sqrt{\rho\sigma r_1}, Bo = \rho g r_1^2/\sigma, We = \rho v_j^2 r_1/\sigma)$ and the criteria for ignoring gravity and viscosity will respectively be $Fr^{-1} = Bo/We << 1$ and $Oh << 1$; the physical reason that $Fr^{-1} << 1$ means that gravity is negligible is the fact that inertia is fighting against gravity to keep the sheet flat rather than capillarity so the $Fr^{-1} << 1$ is a physically more meaningful criteria for ignoring gravity effects compared to $Bo << 1$.

[2] Stating the argument for neglecting the viscous effects by picking the Oh number or the Ca number is considered as the right answer but the most physical one is the $Re << 1$

thus an equal and opposite thrust force will be exerted from the liquid to the disc, i.e. $F_T e_y = -\rho v_j^2(\pi r_j^2) e_y$.

(c):
For finding a relationship for the value of R we have to consider a slice of the rim as our control volume (Figure 3) . The x-component of linear momentum enters the control volume at (1) but vanishes as the drops have zero/negligible velocity in the x-direction. This must happen from the force acting on the fluid in the control volume thus the conservation of linear momentum for the selected C.V. will be:

$$W\rho h(R)U^2 = 2W\sigma$$

substituting $U = u_j$ and $h(R) = r_j^2/2R$ from conservation of mass:

$$\rho u_j^2 r_j^2/(2R) = 2\sigma \Rightarrow \boxed{\frac{R}{r_j} = \frac{\rho u_j^2 r_j}{4\sigma}} \tag{5}$$

Rearranging the result in Equation 5 we can get the following form:

$$\boxed{\frac{R}{r_1} = \frac{\rho u_j^2 r_1}{4\sigma}\left(\frac{r_j}{r_1}\right)^2}$$

which matches well with our predictions from dimensional analysis (equation 3).

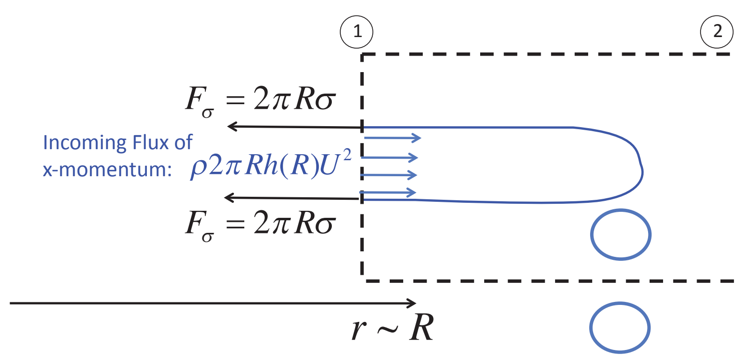

Figure 3: A slice of the sheet rim picked as our control volume (with $h << R$) and length taken to be $W = Rd\theta$.

(d):
Using the boundary layer scaling for diffusion of momentum away from the plate we have the following:

$$\delta \sim \sqrt{\nu t} \sim \sqrt{\nu r_1/u_j}$$

and using the fact that $h_1(inviscid) = r_j^2/2r_1$ we will have:

$$\beta \equiv \frac{\delta}{h_1(inviscid)} = \left(\frac{4r_1^3\nu}{u_j r_j^4}\right)^{1/2} \Rightarrow \boxed{\beta = \frac{2}{\sqrt{Re_j}}\left(\frac{r_1}{r_j}\right)^{3/2}} \tag{6}$$

If we look into β through dimensional analysis then we will have:

$$\beta = f\left(\frac{r_j}{r_1}, Oh \equiv \frac{\mu}{\sqrt{\rho\sigma r_1}}, We \equiv \frac{\rho u_j^2}{(\sigma/r_1)}\right)$$

and by rearranging the equation in (6) we will have:

$$\beta = \frac{Oh^{1/2}}{We^{1/4}}\left(\frac{r_1}{r_j}\right)^{3/2}$$

If viscous effects are important then the momentum of the incoming jet will get dissipated in the boundary layer and thus the liquid sheet emerging from the jet will be slower and thicker.

(e):
At the end of the disc there will be loss of momentum compared to the initial flux from the impinging jet due to the losses in the boundary layer. Through the concept of momentum thickness we know that the loss in the radial momentum at the edge of the disk should be:

$$\Delta MOM = \textit{Loss in the momentum} = 2\rho\pi r_1 \theta u_j^2$$

in which θ is the momentum layer thickness and for simplicity we can assume that θ and δ^* differ at most by a constant of order unity. After the sheet leaves the edge of the disc the velocity in the sheet becomes uniform and conservation of mass and momentum for a C.V. from the impingement point to $r_1 < r$ will give us two equations respectively:

$$\rho\pi r_j^2 u_j^2 - 2\rho\pi r_1 \delta^* u_j^2 = 2\rho\pi r h U^2$$
$$\rho\pi r_j^2 u_j = 2\rho\pi r h U$$

by multiplying the momentum equation by $(1/\rho\pi r_j^2 u_j = 1/2\rho\pi r h U)$ we will find an expression for U and then plugging that into the mass conservation equation we can find another expression for h. We rewrite the mentioned relationships in terms of $\beta = \delta/h_1 = \delta^*/(r_j^2/r_1)$:

$$\boxed{\begin{aligned} &U = u_j(1-\beta) \\ &h = h(r)_{inviscid}/(1-\beta) = (r_j^2/2r)(1-\beta) \end{aligned}} \tag{7}$$

where $h(r)_{inviscid} = r_j^2/2r$ is the variation in the sheet thickness observed in the inviscid case.

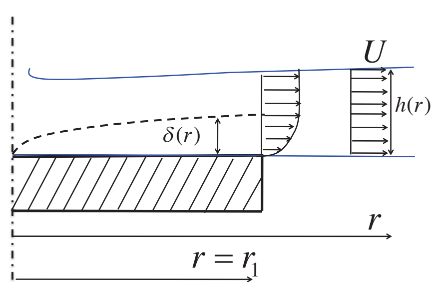

Figure 4: Boundary layer developed on the disc.

Since $\beta > 0$ then we can see that indeed the velocity of the sheet will decrease and the height will increase.

(f):
Using the appendix in either Kundu or Panton we have:

$$\tau_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right)$$

knowing that by axisymmetry the first term on the right hand side is zero (there is no variation in the θ direction):

$$\tau_{\theta\theta} = 2\mu\frac{U}{r}$$

now we can compare the scale of inertial terms with the viscous terms in the liquid sheet:

$$\boxed{\frac{inertia}{viscous} \sim \frac{(1/2)\rho U^2}{2\mu U/r} = \frac{Ur}{4\nu}}$$

if this ratio is larger than unity then viscous terms in the sheet are dominated by inertia terms.

(g):
We can repeat the same argument we had in part (c) but only this time we need to use new expression for U and h using equation (7):

$$\rho W U^2(1-\beta)^2 h_{inviscid}/(1-\beta) = 2\sigma W \Rightarrow \boxed{R/r_j = \rho u_j^2 r_j (1-\beta)/4\sigma} \tag{8}$$

Which rearranging it to the dimensionless form will give:

$$\boxed{\frac{R}{r_1} = \frac{\rho u_j^2 r_1}{4\sigma}\left(\frac{r_j}{r_1}\right)^2 (1-\beta)} \tag{9}$$

using the expression for β we can rewrite the result:

$$\boxed{\frac{R}{r_1} = \frac{\rho u_j^2 r_1}{4\sigma}\left(\frac{r_j}{r_1}\right)^2 \left(1 - \frac{Oh^{1/2}}{We^{1/4}}\left(\frac{r_1}{r_j}\right)^{3/2}\right)} \tag{10}$$

which is exactly in accordance to what we expected from dimensional analysis in the absence of gravitational effects (part a).

(h):
By putting $\lambda_{crit} = R$ and using the result in equation (8) one can get:

$$R = \frac{10\pi\sigma}{\rho_a u_j^2 (1-\beta)^2} \rightarrow \frac{\rho u_j^2 r_j^2 (1-\beta)}{4\sigma} = \frac{10\pi\sigma}{\rho_a u_j^2 (1-\beta)^2} \rightarrow \frac{\rho^2 u_j^4 r_j^2 (1-\beta)^3}{\sigma^2} = \frac{40\pi^2}{\rho_a/\rho}$$

which can be rewritten as:

$$\boxed{We_{crit}(1-\beta)^{3/2} = \sqrt{40\pi/\alpha}} \tag{11}$$

in which $\alpha \equiv \rho_a/\rho$ and $We_{crit} \equiv \rho u_j^2 r_j/\sigma$. In the inviscid case for water in the air ($\rho_a/\rho = 1.2/1000$) we get $We_{crit} \sim 900$. As the effects of viscosity get larger and larger ($\beta \neq 0$) the critical Weber number shifts to higher values (Equation 11) and the rim radius at a given value of the Weber number is reduced, from equation (8). Interestingly because the dependence of equation (11) on β is stronger than equation (8) this means the viscous Savart sheet actually extends a little bit further out radially before it starts flapping (Figure 5).

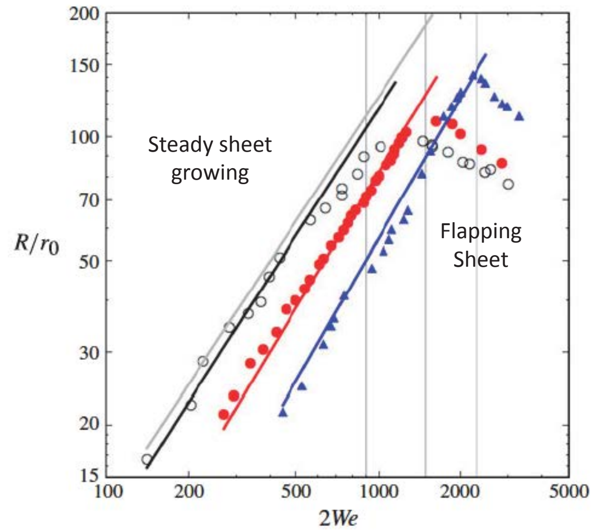

FIGURE 4. (Colour online) Smooth sheet radius for three different viscosities, namely $\nu = \eta/\rho = 10^{-6}\ \mathrm{m^2\ s^{-1}}$ (water, open circles, black), $\nu = 60 \times 10^{-3}/1200\ \mathrm{m^2\ s^{-1}}$ (filled circles, red online), and $\nu = 320 \times 10^{-3}/1200\ \mathrm{m^2\ s^{-1}}$ (filled triangles, blue online). The solid lines are (3.12). The inviscid ($\nu = 0$) reference case for which $R/r_0 = We/4$ is shown as a light grey line: see Villermaux & Clanet (2002) for comparison. The vertical lines indicate the transition

Figure 5: Copy of Figure 4 from Villermaux's paper ([1]).

References

[1] E. Villermaux, V. Pistre, and H. Lhuissier. The viscous Savart sheet. *Journal of Fluid Mechanics*, 730:607–625, August 2013.

2.25 – Fluid Mechanics – Fall 2013

Final Exam, problem 2 solution

Part (a) (1 point)

$$\text{Dimensions of } w: \left[\frac{L^2}{T}\right]$$

$$\text{Dimensions of } U_\infty: \left[\frac{L}{T}\right]$$

$$\text{Dimensions of } a^2 \text{ and } z^2: \left[L^2\right]$$

$$\Rightarrow \left[\frac{L^2}{T}\right] = \left[\frac{L}{T}\right]\left[L^{2\beta}\right]$$

$$\Rightarrow 2\beta = 1 \Rightarrow \boxed{\beta = \frac{1}{2}}$$

Part (b) (1 point)

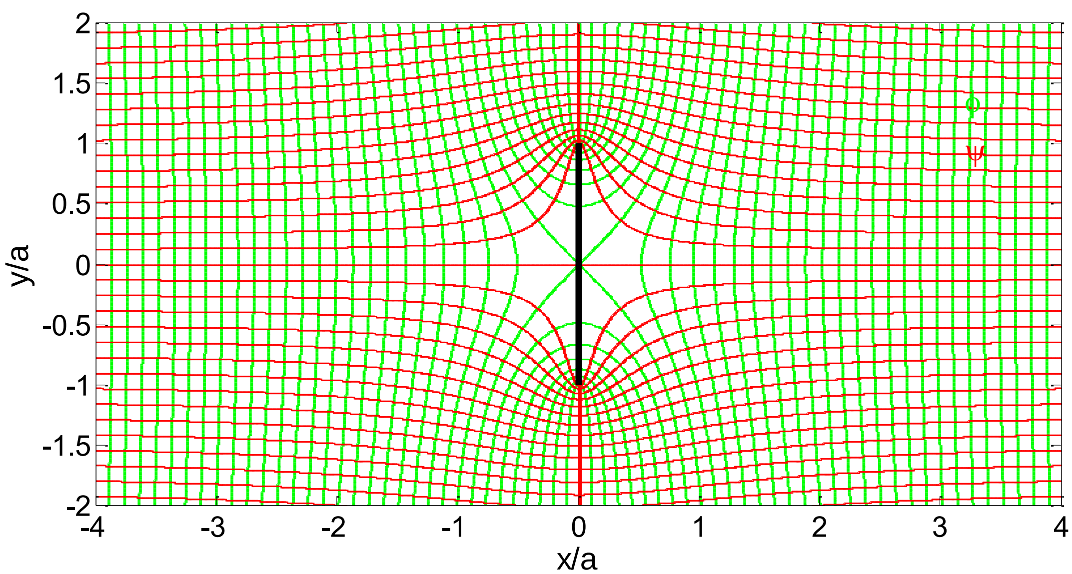

Figure 1: A holiday-themed plot of velocity potential contours (green) and streamlines (red) for potential flow around a flat plate of length 2*a* and negligible thickness. Lines of constant potential and streamlines are everywhere perpendicular, except at the stagnation point at the origin. Flow is from left to right. Note: the vertical "streamlines" at *x* = 0 are not physical; they are artifacts of discontinuities in the stream function at *x* = 0.

Part (c) (3 points)

The complex velocity is

$$\frac{dw}{dz} = v_x - iv_y = \frac{\pm U_\infty z}{\sqrt{z^2 + a^2}} = \frac{\pm U_\infty (x + iy)}{\sqrt{x^2 + 2ixy - y^2 + a^2}}. \tag{1.1}$$

To find v_x along the centerline, set $y = 0$ and isolate the real part of dw / dz:

$$v_x (y = 0) = \frac{\pm U_\infty x}{\sqrt{x^2 + a^2}}. \tag{1.2}$$

Since v_x is positive everywhere (in the reference frame of the plate),

$$\boxed{v_x (y = 0) = \begin{cases} \dfrac{U_\infty x}{\sqrt{x^2 + a^2}}, & x > 0 \\ \dfrac{-U_\infty x}{\sqrt{x^2 + a^2}}, & x < 0 \end{cases}} \tag{1.3}$$

Similarly, the *y*-direction slip velocity on the plate is determined by setting $x = 0$ and looking at the imaginary part of dw / dz:

$$v_y (x = 0) = \frac{\mp U_\infty y}{\sqrt{a^2 - y^2}}. \tag{1.4}$$

Again, use physical intuition to determine the sign of v_y:

$$\boxed{v_y (x = 0, -a < y < a) = \begin{cases} \dfrac{U_\infty y}{\sqrt{a^2 - y^2}}, & x = 0^- \text{ (left side of plate)} \\ \dfrac{-U_\infty y}{\sqrt{a^2 - y^2}}, & x = 0^+ \text{ (right side of plate)} \end{cases}} \tag{1.5}$$

Figure 2 shows the signs of the velocity components in each of the four quadrants.

II $v_x > 0$, $v_y > 0$	I $v_x > 0$, $v_y < 0$
III $v_x > 0$, $v_y < 0$	IV $v_x > 0$, $v_y > 0$

Figure 2: Signs of velocity components in each quadrant.

Part (d) (2 points)

There are at least two curves that may be used to compute the circulation around the top half of the plate.

Curve 1

The simplest curve to use is likely the rectangle ABCD shown below.

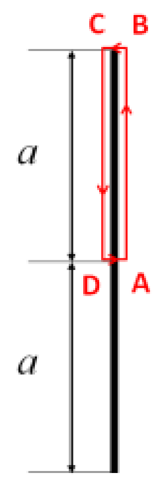

Figure 3: Counter-clockwise curve used for computing circulation.

The circulation is determined from its definition

$$\Gamma = \oint_{ABCD} \mathbf{v}\cdot d\mathbf{r} = \oint_{AB} \mathbf{v}\cdot d\mathbf{r} + \underbrace{\oint_{BC} \mathbf{v}\cdot d\mathbf{r}}_{=0} + \oint_{CD} \mathbf{v}\cdot d\mathbf{r} + \underbrace{\oint_{DA} \mathbf{v}\cdot d\mathbf{r}}_{=0} \tag{1.6}$$

where $\mathbf{v}$ is the velocity vector and $d\mathbf{r}$ is a differential line segment along ABCD. The integrals along BC and DA vanish because the plate is infinitely thin. The integrals along AB and CD reduce to

$$\begin{gathered} \Gamma = \oint_{AB} \mathbf{v}\cdot d\mathbf{r} + \oint_{CD} \mathbf{v}\cdot d\mathbf{r} = \int_0^a \frac{-U_\infty y dy}{\sqrt{a^2-y^2}} + \int_0^a \frac{U_\infty y(-dy)}{\sqrt{a^2-y^2}} \\ \Rightarrow \Gamma = -2U_\infty \int_0^a \frac{y dy}{\sqrt{a^2-y^2}}. \end{gathered} \tag{1.7}$$

This integral may be evaluated by, for example, making the substitution $u = a^2 - y^2,\ du = -2y dy$ to obtain

$$\boxed{\Gamma = -2U_\infty a.} \tag{1.8}$$

Curve II

One can also integrate along a rectangle that includes the entire half-domain $y > 0$. This rectangle extends to infinity in the $+x$, $+y$, and $-x$ directions and coincides with the x-axis. Since the fluid contains no vorticity flux (it is a potential flow, which by definition is irrotational), all of the vorticity within curve II is contained in the top half of the plate. Thus, the answer for circulation should be the same as for curve I.

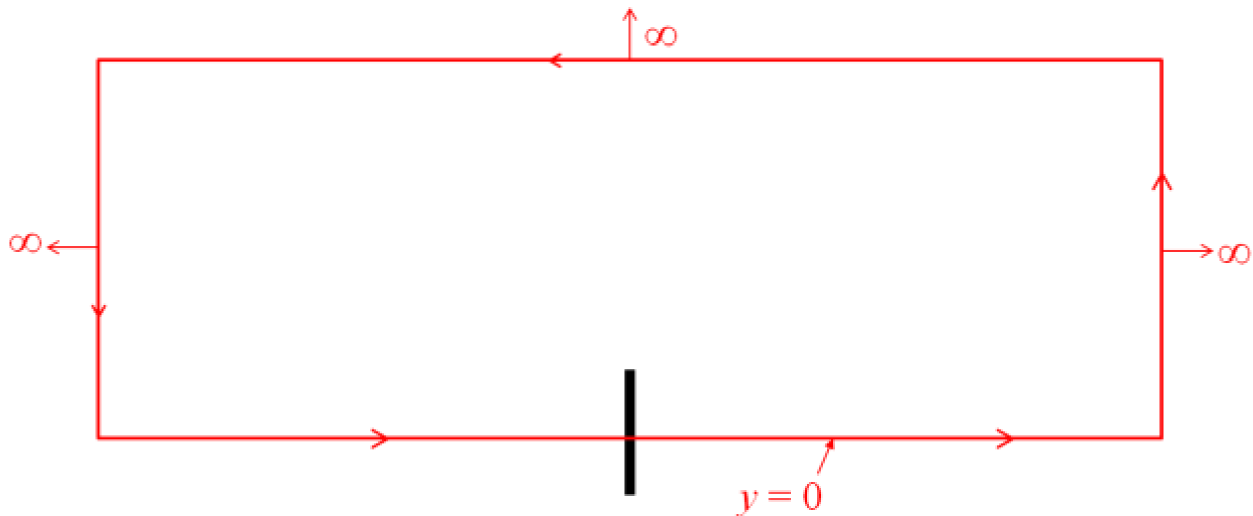

Figure 4: The desired circulation may also be computed by integrating around the entire top half of the domain. The result for circulation is the same because the flow around the plate is irrotational, and so the entire vorticity flux is "bound into" the top half of the plate.

In this case, the y-components of velocity along the left and right segments of the curve are zero (since $\mathbf{v} \to U_\infty \mathbf{e}_x$ as distance from the plate approaches infinity). The circulation becomes

$$\Gamma = \int_{-\infty}^{\infty} -U_\infty dx + \int_{-\infty}^{\infty} U_\infty \frac{|x|}{\sqrt{x^2+a^2}} dx = U_\infty \int_{-\infty}^{\infty} \left(\frac{|x|}{\sqrt{x^2+a^2}} - 1 \right) dx \tag{1.9}$$

By symmetry, this integral can be written

$$\Gamma = 2U_\infty \int_0^{\infty} \left(\frac{x}{\sqrt{x^2+a^2}} - 1 \right) dx = 2U_\infty \lim_{L\to\infty} \left[\sqrt{x^2+a^2} - x \right]_0^L = -2U_\infty a. \tag{1.10}$$

Since the flow is symmetric about the x-axis, the circulation around the bottom half of the plate is equal and opposite to (1.8). Thus the net circulation around the entire plate is zero.

Part (e) (2 points)

It was given that the flow is incompressible (and thus barotropic), inviscid, and free of body forces. If we ignore any flows induced by the removal of the oar, we can thus apply Kelvin's Circulation Theorem and argue that the circulation around the top and bottom halves of the oar is conserved, and the strength of both resulting vortices is given by the circulation found in part (d):

$$\boxed{\Gamma_{bottom} = -\Gamma_{top} = 2U_\infty a.} \tag{1.11}$$

The vorticity may be computed from Stokes' Theorem. Consider the bottom vortex, which has positive circulation:

$$\Gamma_{bottom} = 2U_\infty a = \iint_A \omega_z dA = \omega_z \pi R^2 \Rightarrow \boxed{\omega_{z,bottom} = -\omega_{z,top} = \frac{2U_\infty a}{\pi R^2}.} \tag{1.12}$$

Part (f) (2 points)

Since the vorticity is uniformly distributed in each vortex, the fluid within the vortices is moving in solid-body rotation. The velocity field inside the bottom vortex (ignoring flows induced by the top vortex) is given by

$$v_\theta = \frac{\omega_z}{2} r = \frac{U_\infty a}{\pi R^2} r. \tag{1.13}$$

Evaluating at $r = R$,

$$\boxed{v_\theta\left(r = R\right) = \frac{U_\infty a}{\pi R}.} \tag{1.14}$$

One can also arrive at this answer by assuming the flow field outside the vortex to be that of an irrotational vortex $\left(v_\theta = \Gamma / 2\pi r\right)$, setting $r = R$, and using the result from part (e):

$$v_\theta\left(r = R\right) = \frac{\Gamma}{2\pi R} = \frac{2U_\infty a}{2\pi R} = \frac{U_\infty a}{\pi R}. \tag{1.15}$$

The pressure is obtained from applying the Bernoulli equation between a point at infinity (where the fluid is stationary) and the edge of the vortex. Recall that the Bernoulli equation may be applied between any two points since the flow outside the vortices is irrotational.

Since there are no body forces to consider, the Bernoulli equation reads

$$p_\infty + \underbrace{\frac{1}{2}\rho(0)^2}_{\substack{\text{fluid is stagnant}\\ \text{at infinity in the}\\ \textit{laboratory}\text{ reference}\\ \text{frame}}} = p(r=R) + \frac{1}{2}\rho v_\theta^2(r=R) \tag{1.16}$$

Note that we are considering each vortex to be isolated from the other, i.e. the velocity field induced by the other vortex is ignored in formulating (1.16); we will justify this assumption later. Solving for the desired pressure,

$$\boxed{p(r=R) = p_\infty - \frac{1}{2}\rho\left(\frac{U_\infty a}{\pi R}\right)^2.} \tag{1.17}$$

Part (g) (1 point)

In this part we consider each vortex as an irrotational vortex (i.e., one that can be described by a velocity potential), but now we specifically account for the influence of the other vortex. The velocity field associated with an irrotational vortex is

$$v_\theta = \frac{\pm\Gamma}{2\pi r} = \frac{\pm U_\infty a}{\pi r},\ r > R. \tag{1.18}$$

Each vortex will induce the other one to move. Each vortex core moves at a speed given by evaluating (1.18) at $r = H$:

$$v_\theta(r=H) = \frac{\pm U_\infty a}{\pi H}. \tag{1.19}$$

In the laboratory reference frame, the vortices propagate to the left. Thus,

$$\boxed{V = \frac{U_\infty a}{\pi H}\ \text{from right to left.}} \tag{1.20}$$

Equation (1.20) justifies the assumption made in part (f) that the velocity field induced by the other vortex may be neglected when computing the tangential velocity at the edge of the vortex ($r = R$). From part (f), the tangential velocity at the edge of the vortex scales as $1/R$. Equation (1.20) shows that the contribution to the tangential velocity due to the other vortex will scale as $1/H$. Since $H \gg R$, it is therefore safe to assume that $V \ll v_\theta(r=R)$.

Part (h) (4 points)

The unsteady viscous decay of a free vortex was covered in detail on problem set 8. The governing equation is the Navier-Stokes equation in the θ-direction, which in our case reduces to

$$\frac{\partial v_\theta}{\partial t} = \nu \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r v_\theta\right)\right]. \tag{1.21}$$

The necessary assumptions are that the flow is:

- Axisymmetric $\left(\partial(\cdot)/\partial\theta = 0\right)$
- Unaffected by the velocity field due to the other vortex
- Unidirectional $\left(v_r = v_z = 0\right)$

Initial condition: Initially, the flow field is that of an irrotational vortex:

$$v_\theta\left(r, t=0\right) = \frac{\Gamma}{2\pi r}. \tag{1.22}$$

Here $\Gamma = 2U_\infty a$ is the circulation around the bottom vortex found in part (e).

Boundary conditions (valid for $t > 0$): (1) The velocity is zero in the center of the vortex (similar to solid-body rotation), and (2) very far away the velocity field looks like an irrotational vortex (because the fluid very far away from the vortex core has not yet felt the influence of viscosity):

$$\begin{aligned} v_\theta\left(r=0, t\right) &= 0,\ t>0 \\ v_\theta\left(r\to\infty, t\right) &= \frac{\Gamma}{2\pi r},\quad t>0 \end{aligned} \tag{1.23}$$

Following the problem statement, introduce the dimensionless velocity and similarity variable

$$F = \frac{v_\theta}{\Gamma/2\pi r},\quad \eta = \frac{r^2}{4\nu t}. \tag{1.24}$$

Using the chain rule, we can transform the partial derivatives with respect to t and r in (1.21) to ordinary derivatives with respect to η:

$$\frac{\partial(\cdot)}{\partial t} = \frac{\partial \eta}{\partial t}\frac{d(\cdot)}{d\eta} = \frac{-r^2}{4\nu t^2}\frac{d(\cdot)}{d\eta},$$

$$\frac{\partial(\cdot)}{\partial r} = \frac{\partial \eta}{\partial r}\frac{d(\cdot)}{d\eta} = \frac{r}{2\nu t}\frac{d(\cdot)}{d\eta}.$$

The left and right-hand sides of equation (1.21) are then rewritten as

$$\frac{\partial v_\theta}{\partial t} = \frac{\Gamma}{2\pi r}\frac{-r^2}{4\nu t}\frac{dF}{d\eta},$$

$$\nu\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r v_\theta\right)\right] = \nu\frac{\partial}{\partial r}\left[\frac{\Gamma}{2\pi}\frac{1}{2\nu t}\frac{dF}{d\eta}\right] = \nu\frac{\Gamma}{4\pi\nu t}\frac{r}{2\nu t}\frac{d^2F}{d\eta^2}.$$

The governing equation (1.21) then becomes

$$\frac{-\Gamma r}{8\pi\nu t}\frac{dF}{d\eta} = \frac{\Gamma r}{8\pi\nu t}\frac{d^2F}{d\eta^2} \tag{1.25}$$

$$\Rightarrow \frac{d^2F}{d\eta^2} + \frac{dF}{d\eta} = 0. \tag{1.26}$$

The boundary and initial conditions collapse to

$$\begin{aligned} F\left(\eta = 0\right) &= 0, \\ F\left(\eta \to \infty\right) &= 1. \end{aligned} \tag{1.27}$$

The solution can be found by several different methods. Here, we write the characteristic equation for the ODE (1.26) as

$$s^2 + s = s\left(s+1\right) = 0 \Rightarrow s = 0, -1.$$

Then the general solution is

$$F\left(\eta\right) = A + Be^{-\eta}.$$

From (1.27), we have $A = -B = 1$ and thus

$$F\left(\eta\right) = 1 - e^{-\eta}, \tag{1.28}$$

or, in dimensional terms,

$$\boxed{v_\theta\left(r,t\right) = \frac{U_\infty a}{\pi r}\left[1 - \exp\left(-\frac{r^2}{4\nu t}\right)\right].} \tag{1.29}$$

This result implies that the propagation speed V of the vortices decays with time, according to

$$V\left(t\right) = \frac{U_\infty a}{\pi H}\left[1 - \exp\left(-\frac{H^2}{4\nu t}\right)\right]. \tag{1.30}$$

Part (i) (1 point)

From the solution to part (h), it is straightforward to determine the vorticity distribution as a function of space and time:

$$\omega_z(r,t) = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \underbrace{\frac{1}{r}\frac{\partial v_r}{\partial \theta}}_{=0} = \frac{U_\infty a}{2\pi \nu t}\exp\left(-\frac{r^2}{4\nu t}\right) \equiv \frac{U_\infty a}{2\pi \nu t}\exp\left(-\frac{r^2}{\delta^2}\right), \tag{1.31}$$

The vorticity has a normal distribution centered at the vortex core with standard deviation δ. Note that the peak value (at $r = 0$) changes with time. It is convenient to define the length scale $\boxed{\delta \equiv 2\sqrt{\nu t}.}$

Even without knowing the solution to (h), one can still make an argument based on previous problems we have considered throughout the course that deal with viscous diffusion of vorticity (e.g., Rayleigh problem, boundary layers, etc.). From a similar argument one could conclude that $\boxed{\delta \sim \sqrt{\nu t}.}$

Part (j) (3 points)

The vorticity distributions will begin to overlap when their characteristic widths are approximately equal to $H/2$. This happens at a characteristic time t_c, where

$$\frac{H}{2} \approx \sqrt{\nu t_c} \Rightarrow t_c \sim \frac{H^2}{4\nu}. \tag{1.32}$$

The vortex propagation speed scales with the value found in part (g):

$$V \sim \frac{U_\infty a}{\pi H}. \tag{1.33}$$

Then the distance Δx that the vortices travel before they decay away is roughly the product of V and t_c:

$$\Delta x \approx V t_c \approx \frac{U_\infty a}{\pi H}\frac{H^2}{4\nu} = \underbrace{\left(\frac{U_\infty H}{\nu}\right)}_{\equiv Re_H}\left(\frac{a}{4\pi}\right) \Rightarrow \boxed{\frac{\Delta x}{a} \sim Re_H.} \tag{1.34}$$

A Reynolds number clearly emerges from (1.34). This is not surprising, as the Reynolds number should govern the distance that the vortices propagate before they are spread out by viscosity. Thus, in this case the Reynolds number can be interpreted as a dimensionless propagation distance.