2.25 ADVANCED FLUID MECHANICS   Fall 2013

QUIZ 2

THURSDAY, November 14th, 7:00-9:00 P.M.

OPEN QUIZ WHEN TOLD AT 7:00 PM

THERE ARE TWO PROBLEMS
OF EQUAL WEIGHT

Please answer each question in DIFFERENT books

You may use the course textbook (Kundu or Panton), a binder containing your class notes, recitation problems and TWO pages of handwritten notes summarizing the key equations.

Question 1: Electro-osmotic flow in a capillary tube
Microfluidics concerns fluid flows whose characteristic length scale is on the order of micrometers. One commonly used method for driving flows in micro-devices is to use *electro-osmotic flow* (EOF), which exploits the fact that the walls enclosing the flow typically possess an electric charge, and involves applying an electric field along a microfluidic channel. For this problem, the only thing you need to know about EOF is that, under typical conditions, it results in a *uniform velocity profile*. That is, the no-slip condition is *not* valid at the walls, and the fluid velocity resulting from EOF *does not vary* transverse to the flow direction.
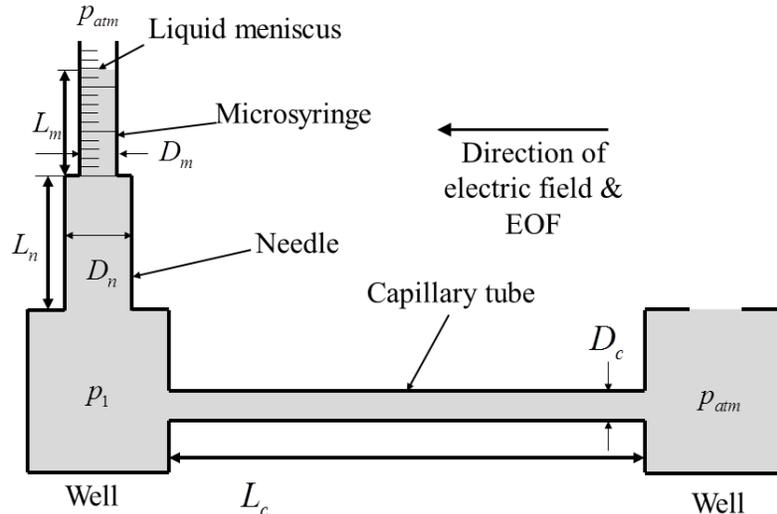
It is important, but not always straightforward, to be able to measure the EOF-driven flow rate in microchannels. One apparatus to do this is depicted in the figure above. A small, cylindrical "capillary tube" with diameter D_c and length L_c connects two wells. The wells and capillary tube are filled with an aqueous solution having density ρ and viscosity μ. An electric field pointing from __right__ to __left__ drives EOF __in the same direction__ at a uniform velocity v_{eo}. The right well is open to the atmosphere and the left well is connected to a needle having diameter D_n and length L_n. The needle feeds into a microsyringe with a smaller diameter D_m, which is open to the atmosphere. L_m is the length of the microsyringe that is filled with fluid. The entire system is kept horizontal to eliminate the effects of gravity.

As fluid accumulates in the left well, over time a pressure difference builds up between the two wells that drives fluid from left to right. This "back-flow" has an *average* velocity given by $\bar{v}_b$. At the same time, fluid is driven upward into the needle and microsyringe, where the average velocities are $\bar{v}_n$ and $\bar{v}_m$, respectively.

In the following, neglect the effects of surface tension (i.e. assume the meniscus at the air/liquid interface in the microsyringe is perfectly horizontal) and assume steady, fully developed, incompressible flow everywhere in the capillary tube.

(a) [2 points] Show that the velocities in the capillary tube, needle, and microsyringe are related by

$$\left(v_{eo} - \bar{v}_b\right)D_c^2 = \bar{v}_n D_n^2 = \bar{v}_m D_m^2. \tag{1}$$

(b) [3 points] Determine the velocity profile in the capillary tube at a point near the middle of the tube (far from either end), in terms of an unknown pressure gradient and $\bar{v}_b$. Clearly

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list your assumptions and boundary conditions. Sketch the velocity profile for two different magnitudes of the pressure gradient (you don't need to specify the values – just one stronger, one weaker).

(c) [2 points] Neglecting entrance effects at the junction between any two pieces of the device, show that

$$\frac{\bar{v}_b L_c}{D_c^2} = \frac{\bar{v}_m L_m}{D_m^2} + \frac{\bar{v}_n L_n}{D_n^2}. \tag{2}$$

The utility of this experimental setup is that it enables one to indirectly measure the electro-osmotic velocity by directly measuring the average meniscus velocity $\bar{v}_m$ and using mathematical manipulations of (1) and (2) to convert this result to v_{eo}. $\bar{v}_m$ is relatively easy to measure by fixing a camera on the microsyringe and tracking the position of the meniscus over time.

(d) [1 point] Show that the needle and microsyringe can be lumped together as a single tube with average fluid velocity $\bar{v}_m$ and diameter D_m if we assume its effective length is

$$L_{eff} = L_m \left[1 + \left(L_n / L_m \right) \left(D_m / D_n \right)^4 \right].$$

(e) [3 points] Show that the electro-osmotic velocity is related to the average meniscus velocity and the average pressure-driven back-flow velocity by

$$\frac{\bar{v}_m}{v_{eo}} = \frac{1}{\left(D_m / D_c \right)^2 + \left(D_c / D_m \right)^2 \left(L_{eff} / L_c \right)}, \tag{3}$$

$$\frac{\bar{v}_b}{v_{eo}} = \frac{1}{1 + \left(D_m / D_c \right)^4 \left(L_c / L_{eff} \right)}. \tag{4}$$

(f) [2 points] Determine the maximum pressure difference that the capillary can support, in terms of the electro-osmotic flow velocity v_{eo}.

(g) [2 points] How does the meniscus height scale with time at short times? That is, find n in the expression $L_{eff} \propto t^n$. Assume here that the situation is quasi-steady, i.e. that everything you have derived above for the steady case is still valid. There is no need to use the unsteady momentum or the unsteady Bernoulli equation.

Question 2: Flow of Paint in a 'Bell Atomizer'
During modern car production, the application of paint is carried out robotically. A computer-controlled robot arm is programmed to uniformly apply paint to all of the internal and external surfaces with a minimum of "overspray". This requires a unique nozzle that can handle large volumes of paints of a range of viscosities without clogging or blocking. The dominant technology has become the *rotary bell atomizer* shown in the Figure below.
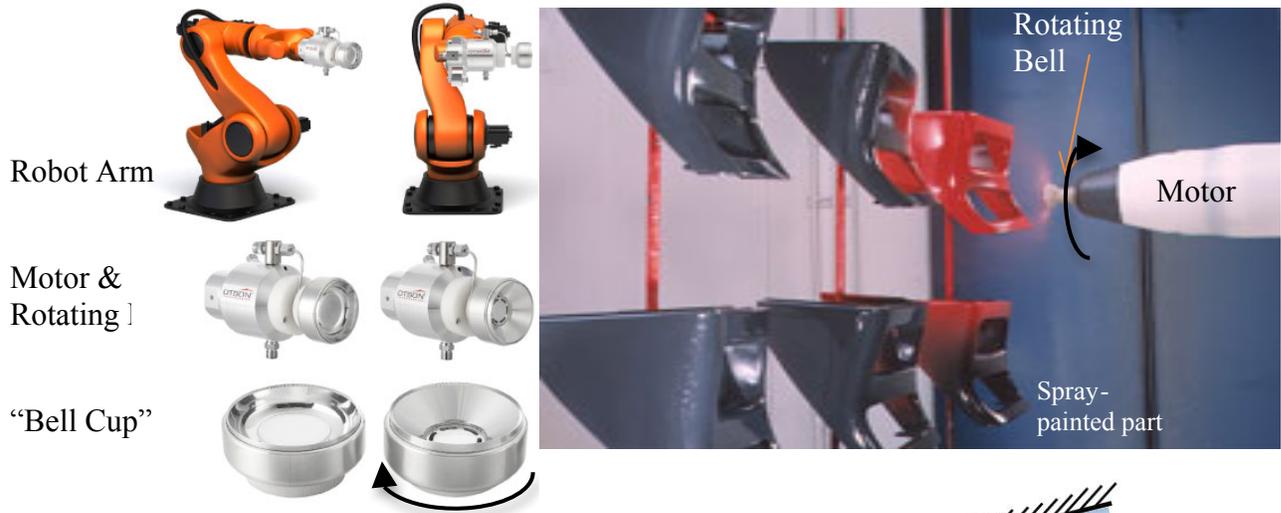
Viscous liquid of viscosity μ, density ρ and surface tension σ is ejected through a hole (at the center of the axis of rotation and at a constant volumetric flow rate Q) onto a smooth conical surface or "bell" (with angle $\theta_0 < 1$) that is rapidly rotating at a constant angular velocity Ω. We describe the flow in a spherical polar coordinate system $\{r, \theta, \phi\}$ (see figure opposite). Centrifugal force holds the fluid film against the wall of the "bell" and drives the fluid radially outwards towards the exit lip at $r = R$. The fluid film thickness $h(r)$ is always very thin compared to the effective radius of the cup $\xi = r\sin\theta_0$ at each point throughout the device and forms a thin and very uniform coating by the time it reaches the exit of the applicator.

We will represent the free surface height as $h(r)$ and the radial velocity by the function $v_r(r,\theta)$. You can neglect the density and viscosity of air as well as the effects of gravitational acceleration and pressure gradients in the film (these are both overwhelmed by the centripetal acceleration of the fluid in the rapidly spinning bell).

a) [3 points] This is a complicated problem with many possible control parameters for a plant operator, or robot path planner or paint formulator to vary. A common language is needed to prepare suitable applicator charts and design new bell applicators. This can best be done using the tools of dimensional analysis. Identify the appropriate dimensionless groups that control the thickness h_R of the paint film at the exit of the applicator ($r = R$). Select both

the density and surface tension of the paint as two of your primary variables.

b) [2 points] Write down the conservation of mass in spherical polar coordinates and show that a velocity vector $\mathbf{v} = [v_r(r,\theta), v_\theta(r,\theta), \Omega r \sin\theta_0]$ consisting of a two-dimensional axisymmetric velocity profile in the film, combined with a constant angular velocity throughout the film (at each value of r) $v_\phi = \Omega r \sin\theta_0$ is able to satisfy the continuity equation. Make a scaling argument to determine if v_θ is small or large compared to v_r ?

c) [4 points] Now consider the radial component of the conservation of momentum equation for the paint film on the rapidly-spinning conical applicator shown in the figure above. Because the layer is 'thin' (so that the value of θ doesn't vary much), it can be argued that it is valid to simplify the angular dependence of the radial velocity component, so that $\sin\theta \approx \sin\theta_0$ everywhere. Show that the radial component of the equation of motion can be simplified to the following form for a very viscous thin film of paint:

$$\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2} + \frac{\rho\Omega^2 r}{\mu}\sin^2\theta_0 \approx 0 \qquad (2.1)$$

Give three appropriate constraints that identify when this approximation is valid.

d) [1 point] Give appropriate boundary conditions at the free surface between the paint and the air and at the rigid wall between the cup and the paint.
[Hint: To do this, it is easiest to sit in the rapidly and steady rotating frame of the cup with angular velocity $v_\phi/r\sin\theta_0 = \Omega$; you can then focus just on the radial (outwards) velocity towards the edge of the bell].

e) [2 points] Find an expression for the radial velocity profile in the fluid film.
[Hint: you may find it useful to define a new coordinate as shown in the figure with $y = r\sin(\theta_0 - \theta)$ and also recall that the function $\sin x \approx x + O(x^3)$ for $x \ll 1$.

f) [2 points] Integrate your expression for the velocity field through each small annular slice of area $dA = 2\pi(r\sin\theta_0)dy$ to show that the constant volume flow rate Q of paint and the thickness $h(r)$ of the paint film are related by the expression

$$h(r) = \left(\frac{3\mu Q}{2\pi\rho\Omega^2 r^2 \sin^3\theta_0}\right)^{1/3} \qquad (2.2)$$

g) [1 point] In the laboratory reference frame, fluid streamlines and particle pathlines actually follow *spiral trajectories* as the paint flows outwards of the conical bell cup, because of the combined radial and azimuthal (i.e. rotational) flow components. Use the definition of a streamline to show that the expression for the spiral trajectory of a material point $r_s(\phi)$ (such as a small bubble or particle) that is on the paint surface is given by:

$$\frac{dr_s}{d\phi} = A r_s^{-1/3} \qquad (2.3)$$

and give an expression for the constant A.

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