

**2.25 ADVANCED FLUID MECHANICS      Fall 2013**

FINAL

TUESDAY, December 17th, 9:00-12:00 P.M.

**OPEN QUIZ WHEN TOLD AT 9:00 AM**

THERE ARE TWO LONG PROBLEMS  
(worth 15 points and 20 points respectively)

Plus 5 “Conceptual Questions” (worth one point each)

Please answer each long question in DIFFERENT books

ALSO: Please write your name on page 2 with Concept Questions  
AND HAND THIS EXAM SCRIPT IN WITH THE BOOK  
CONTAINING QUESTION 1

**You may use the course textbook (Kundu or Panton), a binder containing your class notes, recitation problems and THREE pages of handwritten notes summarizing the key equations.**

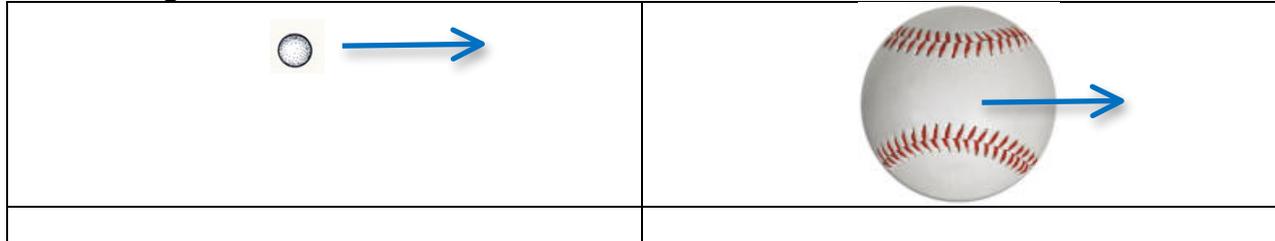


**Family Name:**

**First Name:**

**Concept Question [5 Points]:**

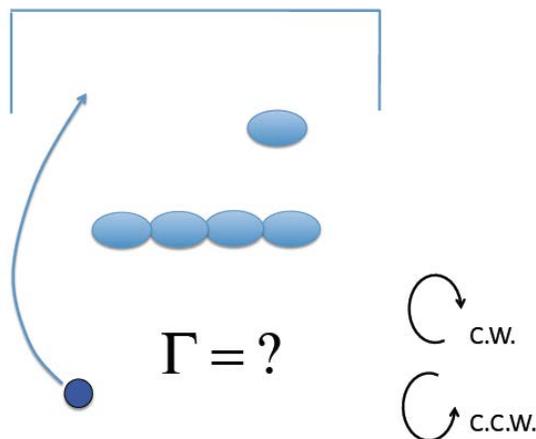
1. A single spherical bacterium moving in water vs. a baseball moving in air which one has a higher  $C_D$ ?



2. For each flow description on the left, choose the form of mass conservation you would use for that situation on the right.
 

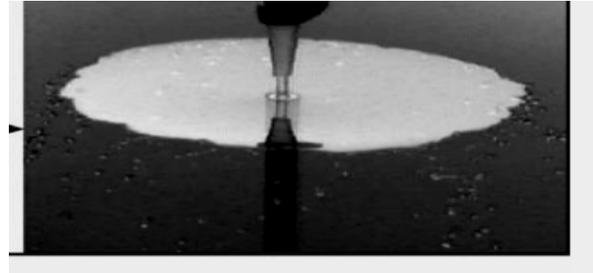
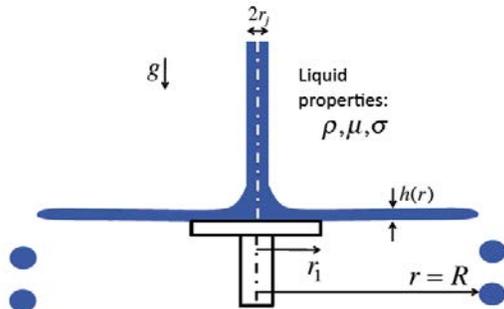
a. Steady/compressible flow of air	1. $\nabla \cdot \vec{u} = 0$
b. Arbitrary flow (Lagrangian perspective)	2. $D\rho / Dt = -\nabla \cdot \vec{u}$
c. Unsteady/incompressible flow of viscous oil	3. $\partial\rho / \partial t + \nabla \cdot (\rho\vec{u}) = 0$
d. Arbitrary flow (Eulerian perspective)	4. $\nabla \cdot (\rho\vec{u}) = 0$
3. How would the three phase contact angle (between liquid/air/solid) ( $\theta_E$ ) change with respect to the Bond number for an ideally smooth surface?
 

a. It will decrease	b. It will increase	c. It remains constant
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4. Write down in one line the scaling for the pressure in a lubrication problem in a confined geometry with characteristic length (L) and height (h) viscosity ( $\mu$ ) and velocity (V) (assumptions  $h \ll L$ )?
5. A soccer player is trying to score a goal, but the players from the other team are blocking his way. So, he decides to curve the ball. What direction of circulation will make the ball go into the goal?



### 1. Savart Liquid Sheets [15 Points]

Atomization processes in rocket engines and electronics cooling applications frequently use an *Impinging Jet geometry* as shown in Figure 1 below. Viscous liquid of viscosity  $\mu$ , density  $\rho$  and surface tension  $\sigma$  is fired vertically downwards out of a nozzle at a constant velocity  $u_j$  as a cylindrical jet of radius  $r_j$  and hits a small rigid circular disk of radius  $r_1$  that is aligned to be orthogonal to the jet. The impact leads to a rapid deflection of the jet and rearrangement of the fluid so that a thin circular sheet of fluid of thickness  $h < r_j$  flows outwards radially.



A few important experimental observations that can be made are:

- i) for small jet diameters and high speeds the sheet remains almost perfectly horizontal and doesn't sag under gravity.
- ii) the circular disk of fluid flows outwards to a critical radial point  $R$  and then abruptly stops and forms a 'rim' that subsequently breaks into small drops that fall vertically downwards under gravity.
- iii) experimental measurements show that small seed particles or bubbles move outwards along this liquid sheet with a constant velocity  $U$  (so that their position varies as  $r_p = Ut$  for  $r_1 \leq r \leq R$ )

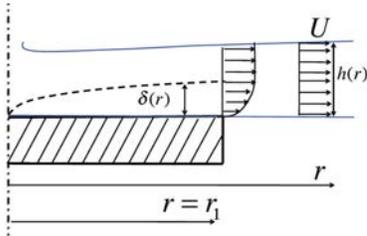
We describe the flow in a cylindrical polar coordinate system  $r, \theta, z$ . We will consider two cases; firstly one in which fluid viscosity is negligible and then one in which viscous effects near the disk do play a modifying role.

- a) [2 points] We seek to understand how the sheet radius  $R$  varies with the incoming momentum of the jet and the size of the disk  $r_1$ . Using a suitable dimensionless analysis and selecting the surface tension as the relevant fluid property, identify the key groups that control the size of the fluid sheet.
  - Based on your analysis, give two suitable constraints that would enable you to tell people when the following conditions hold:
    - i) gravitational effects on the circular sheet are negligible so that sagging doesn't occur
    - ii) viscous effects in the film as it first flows radially outwards over the disk are negligible
- b) [2 points] Show that the thickness of the fluid film flowing radially outwards is  $h_1 = r_j^2 / 2r_1$ . Give an expression for the steady thrust force (including sign) that acts on the solid disk.
- c) [2 points] In this part we neglect any effects of viscosity:
  - Given the experimental observation that the radial velocity is constant in the outward flowing film (until it reaches the rim) find an expression for the thickness of the fluid film  $h(r)$  and then

by considering a radial momentum balance find an expression for the rim radius  $R$ . Arrange your results into a dimensionless form; i.e.  $R/r_1 = ??$

- Describe in words what forces (or stresses) physically balance each other at this point?

- d) [2 points] When the fluid viscosity is small, but not completely negligible, a viscous boundary layer forms near the disk and grows as the fluid flows radially outwards. Give an appropriate expression for the thickness of the boundary layer at the edge of the disk (where  $r \rightarrow r_1$ ).



To express this in dimensionless form we compare this value to the thickness of the inviscid sheet and call the ratio  $\beta = \delta/h_1 = ??$

- If viscous effects are important, do you expect the fluid film to be thicker and moving slower, or thinner and moving faster than in the inviscid case?

- e) [2 points] Use the concept of a *displacement thickness* to give an expression for the momentum loss in the viscous sheet as it exits the disk radially at  $r = r_1$ . Then using the conservation laws find expressions for the average velocity and height of the sheet just as the fluid leaves the disk. Check that your result is in agreement with the conclusion you reached above in (d).
- f) [2 points] Beyond the edge of the disk (i.e. for radii  $r_1 \leq r \leq R$ ) there is no rigid surface to provide additional shearing, and the thin but viscous fluid film quickly returns to a plug-like profile in which  $U$  is constant throughout the fluid sheet. Use the results of your momentum balance from part (c) to evaluate an expression for the viscous stress  $\tau_{\theta\theta}$ . Give a dimensionless condition for when viscous effects are small (but not totally negligible) in the sheet.
- g) [2 points] Use your expressions from parts (d) and (e) to construct a new expression for the radius of the sheet at which the radial outward flow stops;  $R/r_1 = ??$  (hint your result should reduce correctly to the case when fluid viscosity is totally negligible!)
- h) [1 point] Experiments show that beyond a critical flow rate the disk transitions to a time-periodic ‘flapping mode’ (reminiscent of a flapping flag or a pulsating jellyfish). A linear stability analysis (like the one described in the last class for a fluid jet) gives the following critical condition for the critical wavelength  $\lambda$  of the instability:

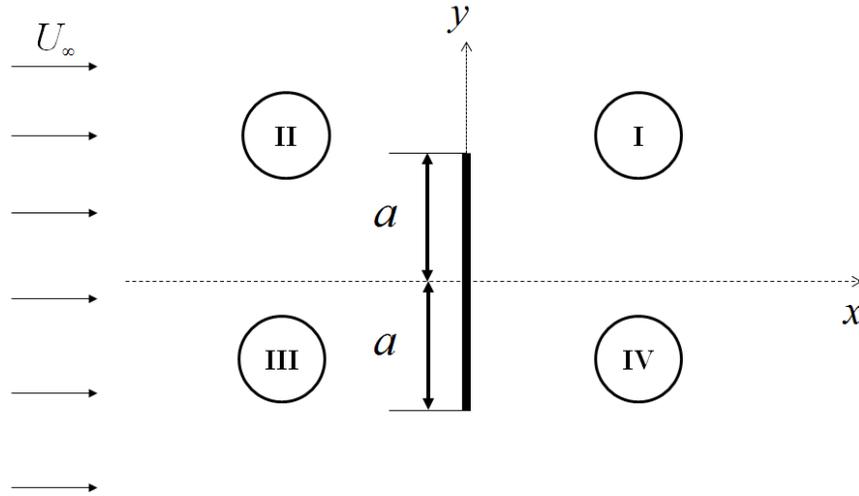
$$\lambda_{crit} = 10\pi \frac{\sigma}{\rho_a U^2} \quad (\text{where } \rho_a \text{ is the density of the air surrounding the fluid sheet})$$

Assuming that the sheet starts flapping when it extends radially outwards to be big enough to fit in exactly one flapping wavelength (i.e. so the sheet radius grows to be  $R \rightarrow \lambda_{crit}$ ), combine the expression above with your expression for the radius of a viscous sheet to give a dimensionless expression for the critical Weber number  $We_{crit}$  at onset of the flapping instability.

## 2. Flow Around an Oar [20 points]

To study the flow around a rowing oar, we consider a simpler but related two-dimensional problem. The problem involves a vertical thin plate, height  $2a$  and negligible thickness, moving in an unbounded fluid with horizontal velocity  $U_\infty$  perpendicular to its long axis. The fluid density is  $\rho$  and the kinematic viscosity is  $\nu$ .

Throughout the problem, you may neglect all body forces.



**Figure 1: Potential flow over an oar of length  $2a$  and negligible thickness oriented perpendicular to the free-stream flow direction. This figure is shown in the reference frame of the plate. Roman numerals indicate 1<sup>st</sup>-4<sup>th</sup> quadrants.**

### Part I: Potential flow

Initially, we assume that the flow has no separation around the plate and that potential flow theory is valid. The complex potential for this flow (in the reference frame of the plate) is known to be

$$w(z) = \phi + i\psi = \pm U_\infty (z^2 + a^2)^\beta \quad (1)$$

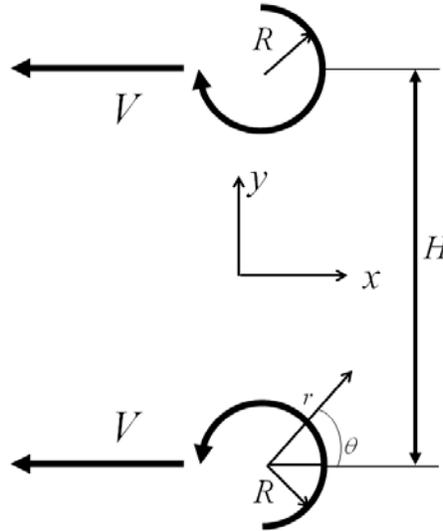
where  $z = x + iy$ ,  $\beta$  is a positive constant ( $\beta \neq 1$ ) and  $2a$  is the height of the plate.

- [1 point] Use basic dimensional arguments to determine the value of  $\beta$ .
- [1 point] Draw 5 representative streamlines and 5 lines of constant velocity potential around the plate.
- [3 points] It can be shown that the complex velocity  $\frac{dw}{dz}$  is given by

$$\frac{dw}{dz} = v_x - iv_y \quad (2)$$

Use this definition to determine the  $y$ -direction velocity on the surface of the plate ( $x = 0$ ,  $|y| < a$ ) and the  $x$ -direction velocity on the centerline ( $y = 0$ ). Carefully note the sign of these velocities! Use physical arguments to determine the sign of the  $x$ - and  $y$ -components of velocity (in the reference frame of the plate) in each quadrant (I-IV).

- (d) [2 points] Determine the circulation (sign and magnitude, assuming positive circulation is counter-clockwise) around (i) the top half of the plate (i.e., the portion for which  $y > 0$ ) and (ii) the entire plate. Clearly indicate the contours used in computing the circulation.



**Figure 2: Vortex pair left in the fluid after sudden removal of the plate. This figure is in the laboratory reference frame (the bulk fluid is stationary).**

### Part II: Vortex propagation

Now consider the situation when the oar is suddenly removed from the fluid. The resulting flow can be described as a pair of vortices a vertical distance  $H$  apart, moving with a forward speed  $V$ . The fluid outside of the vortices is irrotational. The vorticity distribution within each vortex is assumed to be uniform over a radial distance  $R$  from the center. Assume  $H \gg R$ .

- (e) [2 points] Neglecting any flows induced by the removal of the oar, what is the strength (i.e. the magnitude of circulation) of each vortex? Justify your answer, stating any necessary assumptions. What is the magnitude of the vorticity in each of the vortices?
- (f) [2 points] Determine (i) the tangential velocity and (ii) the pressure at the “edge” of the bottom vortex, at  $r = R$  (where  $r$  is the distance outward from the center – see Figure 2), in terms of the circulation found in part (e).
- (g) [1 point] Assuming potential flow, calculate the propagation speed  $V$  of the vortex pair (it should depend on  $U_\infty$ ).

### Part III: Vorticity diffusion

In reality, viscous stresses will diffuse the vorticity outward. Let us focus our attention on the flow field around one of the vortices. Initially, the vortex resembles an ideal vortex ( $R \rightarrow 0$ ). As time passes, the flow resembles solid-body rotation near the center and an ideal vortex far from the center.

- (h) [4 points] Derive the governing equations, initial condition, and boundary conditions for this unsteady process. Use the similarity transformation below to show that the governing equation reduces to

$$\frac{d^2 F}{d\eta^2} + \frac{dF}{d\eta} = 0, \quad (3)$$

subject to appropriate boundary conditions. Here  $F = \frac{v_\theta}{\Gamma / 2\pi r}$  is a dimensionless velocity and  $\eta = \frac{r^2}{4\nu t}$  is a dimensionless similarity variable. Solve equation (3) and thus determine the velocity profile around one of the vortices. You may find the following identity helpful: if  $dy/dx = (dy/du)(du/dx)$ ,

$$\frac{d^2 y}{dx^2} = \frac{d^2 y}{du^2} \left( \frac{du}{dx} \right)^2 + \frac{dy}{du} \left( \frac{d^2 u}{dx^2} \right).$$

- (i) [1 point] Estimate a characteristic width (in the  $y$  direction) of the vorticity distribution. It should depend on time.
- (j) [3 points] Estimate the time required for the vorticity distributions associated with each vortex to overlap. Use your result to estimate the distance traveled by the vortices before they overlap. What dimensionless parameter appears? Explain why this answer makes sense.

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