

MIT Department of Mechanical Engineering
2.25 Advanced Fluid Mechanics

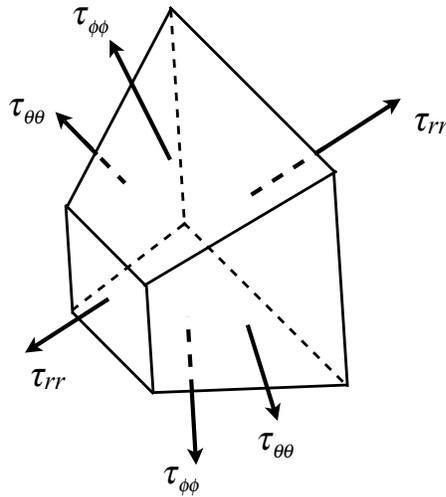
Problem 8.13

This problem is from “Advanced Fluid Mechanics Problems” by A.H. Shapiro and A.A. Sonin

Consider a gas bubble of fixed mass and radius $R(t)$ which is expanding or contracting in an infinite sea of incompressible liquid. The speed of the interface is dR/dt . The local Eulerian coordinate in the liquid is r . Let p_R , p , and p_∞ be, respectively the pressure at $r = R$ (on the liquid side of the interface), at $r = r$, and at $r = \infty$.

- (a) Determine the viscous contribution to the normal stress τ_{rr} in the liquid.
- (b) Show that the dimensionless overpressure, $(p_R - p_\infty)/\rho(dR/dt)^2$, is independent of whether the fluid is viscous or inviscid.

Solution:



- (a) First, we must determine the velocity field in the liquid at any point in time. We choose a control volume taking the shape of a hollow sphere with inner control surface at radius $R(t)$, which moves outward at exactly the rate of expansion of the bubble dR/dt , and outer surface at an arbitrary radius r , such that its volume is $V = \frac{4}{3}\pi(r^3 - R(t)^3)$. Using Form A of the integral mass conservation equation,

$$\frac{d}{dt} \int_{CV} \rho dV + \int_{CS} \rho(\mathbf{u} - \mathbf{u}_{CS}) \cdot \hat{n} dA = 0 \quad (8.13a)$$

we solve for the radial velocity u_r at any position r

$$u_r = \frac{R^2}{r^2} \frac{dR}{dt} \quad (8.13b)$$

Using Eq. (8.13b) we can determine the average rate of strain from the following equations

$$\dot{\gamma}_{rr} = 2 \frac{\partial u_r}{\partial r} \quad (8.13c)$$

and

$$\dot{\gamma}_{\theta\theta} = 2 \left(\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right) \quad (8.13d)$$

and

$$\dot{\gamma}_{\phi\phi} = 2 \left(\frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta \cot \theta}{r} \right) \quad (8.13e)$$

There is no azimuthal or polar velocity in this flow, $u_\theta = u_\phi = 0$, and hence

$$\dot{\gamma}_{rr} = -4 \frac{R^2}{r^3} \frac{dR}{dt} \quad (8.13f)$$

and

$$\dot{\gamma}_{\theta\theta} = 2 \frac{R^2}{r^3} \frac{dR}{dt} \quad (8.13g)$$

and

$$\dot{\gamma}_{\phi\phi} = 2 \frac{R^2}{r^3} \frac{dR}{dt} \quad (8.13h)$$

For a Newtonian fluid, $\tau_{ij} = \mu \dot{\gamma}_{ij}$, where μ is the dynamic viscosity. Accordingly the normal stresses for this flow are

$$\tau_{rr} = -4\mu \frac{R^2}{r^3} \frac{dR}{dt} \quad (8.13i)$$

and

$$\tau_{\theta\theta} = 2\mu \frac{R^2}{r^3} \frac{dR}{dt} \quad (8.13j)$$

and

$$\tau_{\phi\phi} = 2\mu \frac{R^2}{r^3} \frac{dR}{dt} \quad (8.13k)$$

(b) The complete equation of motion in the radial direction for spherical coordinates is

$$\rho \left(\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} + \frac{u_\phi}{r \sin \theta} \frac{\partial u_r}{\partial \phi} - \frac{u_\theta^2 + u_\phi^2}{r} \right) \quad (8.13l)$$

$$= \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{rr}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\tau_{\theta r} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial \tau_{\phi r}}{\partial \phi} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r} \right] - \frac{\partial p}{\partial r} + \rho g_r \quad (8.13m)$$

Neglecting gravity and retaining only the non-zero terms, we have

$$\rho \left(\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} \right) = \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{rr}) + \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r} \right] - \frac{\partial p}{\partial r} \quad (8.13n)$$

Substituting Eq. (8.13i), (8.13j) and (8.13k) into Eq. (8.13n) to evaluate the net contribution of viscous stresses acting on a fluid element, we obtain

$$\rho \left(\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} \right) = \left[-\frac{4\mu}{r^2} \frac{\partial}{\partial r} \left(\frac{R^2}{r} \frac{dR}{dt} \right) - 4\mu \frac{R^2}{r^4} \frac{dR}{dt} \right] - \frac{\partial p}{\partial r} \quad (8.13o)$$

When we differentiate this term, we find that the net contribution of viscous stresses acting radially is exactly zero and hence there is no net dissipation associated with this flow. Consequently, the governing equation is

$$\rho \left(\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} \right) + \frac{\partial p}{\partial r} = 0 \quad (8.13p)$$

Integrating Eq. (8.13p) along dr , which is indeed the streamline coordinate, we obtain the unsteady Bernoulli equation

$$\rho \int_{r=R}^{r=\infty} \frac{\partial u_r}{\partial t} dr' + \rho \int_{u_r=\frac{dR}{dt}}^{u_{r,\infty}=0} u_r' du_r' + \int_{p_R}^{p_\infty} dp' = 0 \quad (8.13q)$$

which is

$$\rho \int_{r=R}^{r=\infty} \frac{R^2}{r'^2} \frac{d^2 R}{dt^2} dr' - \frac{1}{2} \rho \left(\frac{dR}{dt} \right)^2 + p_\infty - p_R = 0 \quad (8.13r)$$

which gives

$$\rho R \frac{d^2 R}{dr^2} - \frac{1}{2} \rho \left(\frac{dR}{dt} \right)^2 + p_\infty - p_R = 0 \quad (8.13s)$$

which at last yields the final result

$$\frac{p_R - p_\infty}{\frac{1}{2} \rho \left(\frac{dR}{dt} \right)^2} = \frac{2R \frac{d^2 R}{dt^2}}{\left(\frac{dR}{dt} \right)^2} - 1 \quad (8.13t)$$

So we have shown that the dimensionless overpressure is indeed independent of whether the fluid is viscous or inviscid. Note that the dimensionless overpressure can be positive or negative depending on the rate of change of the surface velocity of the gas bubble.

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