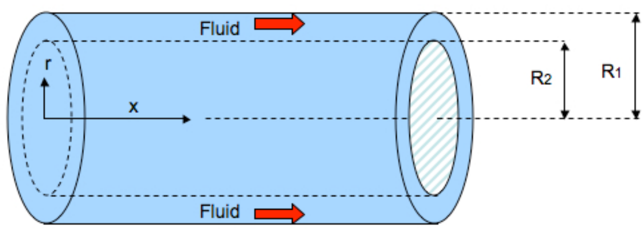
MIT Department of Mechanical Engineering

2.25 Advanced Fluid Mechanics

Problem 6.04a

This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin



Consider a steady, fully developed laminar flow in an annulus with inside radius R_2 and outside radius R_1.

- (a) Find a relation between the pressure gradient $\frac{dp}{dx}$, the volume flow rate Q, the fluid viscosity μ, R_1, and $\frac{R_2}{R_1}$.
- (b) Fin the limiting form of the relation for a very thin annulus by expressing it in terms of R_1 and $\frac{h}{R_1}$, where $h = R_1 - R_2$, and taking the limit $\frac{h}{R_1} \to 0$. Compare with the formula for fully developed laminar flow between parallel flat plates separated by a distance h.
- (c) In the opposite limit $\frac{R_2}{R_1} \to 0$, does the relation of (a) reduce to the formula for Hagen-Poiseuille flow in a circular pipe of radius R_1? Discuss your answer.

Solution:

- From the N-S in cylindrical coordinates, the equation can be reduced to

$$0 = -\frac{1}{\mu}\frac{\partial p}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}\right), \tag{6.04aa}$$

where the first term is approximately a constant across the space between the cylinders (long cylinder approximation), then

$$0 = -K + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}\right), \tag{6.04ab}$$

then, integrating,

$$\int Krdr = \int \frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}\right)dr, \quad \Rightarrow \quad K\frac{r^2}{2} + C_1 = r\frac{\partial v_x}{\partial r}, \quad \Rightarrow \quad K\frac{r}{2} + \frac{C_1}{r} = \frac{\partial v_x}{\partial r}. \tag{6.04ac}$$

Now, integrating again

$$\int (K\frac{r}{2} + \frac{C_1}{r})dr = \int (\frac{\partial v_x}{\partial r})dr, \quad \Rightarrow \quad K\frac{r^2}{4} + C_1\ln(r) + C_2 = v_x, \tag{6.04ad}$$

Then, applying the boundary conditions,

$$v_x(R_1) = 0, v_x(R_2) = 0, \tag{6.04ae}$$

the constants can be obtained. Then,

$$K\frac{R_1^2}{4} + C_1\ln(R_1) + C_2 = 0, \quad OR \quad K\frac{R_2^2}{4} + C_1\ln(R_2) + C_2 = 0. \tag{6.04af}$$

Now, substracting the solutions to obtain C_1,

$$\frac{K}{4}(R_1^2 - R_2^2) + C_1\ln\frac{R_1}{R_2} = 0, \tag{6.04ag}$$

then,

$$C_1 = -\frac{\frac{K}{4}(R_1^2 - R_2^2)}{\ln\frac{R_1}{R_2}}. \tag{6.04ah}$$

Now, re-expressing in terms of the requested variables,

$$C_1 = -R_1^2\frac{\frac{K}{4}(1 - \Phi^2)}{-\ln\Phi}, \quad \Rightarrow \quad C_1 = -R_1^2\frac{\frac{K}{4}(\Phi^2 - 1)}{\ln\Phi}, \tag{6.04ai}$$

where, $\Phi = R_2/R_1$.
Now, for C_2, we can use any of the two equations,

$$C_2 = -K\frac{R_1^2}{4} - C_1\ln(R_1), \qquad C_2 = -K\frac{R_2^2}{4} - C_1\ln(R_2). \tag{6.04aj}$$

Upon substitution of C_1,

$$C_2 = -K\frac{R_1^2}{4} + \frac{\frac{K}{4}(R_1^2 - R_2^2)}{\ln\frac{R_1}{R_2}}\ln(R_1), \qquad C_2 = -K\frac{R_2^2}{4} + \frac{\frac{K}{4}(R_1^2 - R_2^2)}{\ln\frac{R_1}{R_2}}\ln(R_2), \tag{6.04ak}$$

simplifying,

$$C_2 = \frac{K}{4}\left(-R_1^2 + \frac{(R_1^2 - R_2^2)}{\ln\frac{R_1}{R_2}}\ln(R_1)\right), \tag{6.04al}$$

$$C_2 = \frac{K}{4}\left(-R_2^2 + \frac{(R_1^2 - R_2^2)}{\ln\frac{R_1}{R_2}}\ln(R_2)\right). \tag{6.04am}$$

Then the velocity is

$$v_x = \frac{1}{4\mu}\frac{dp}{dx}\left[r^2 - R_2^2 - \frac{R_1^2 - R_2^2}{\ln(R_1/R_2)}\ln\left(\frac{r}{R_2}\right)\right] \tag{6.04an}$$

Now, to obtain the flux, let's integrate this expression,

$$\int_0^{2\pi}\int_{R_2}^{R_1} v_x r dr d\theta = \int_0^{2\pi}\int_{R_2}^{R_1}\left(K\frac{r^2}{4} + C_1\ln(r) + C_2\right) r dr d\theta, \tag{6.04ao}$$

$$\int_0^{2\pi}\int_{R_2}^{R_1}\left(K\frac{r^2}{4} + C_1\ln(r) + C_2\right) r dr d\theta = 2\pi\int_{R_2}^{R_1}\left(K\frac{r^3}{4} + C_1 r\ln(r) + C_2 r\right) dr \tag{6.04ap}$$

After integration,

$$2\pi\int_{R_2}^{R_1}\left(K\frac{r^3}{4} + C_1 r\ln(r) + C_2 r\right) dr = 2\pi\left(\frac{Kr^4}{16} + C_2\frac{r^2}{2} + C_1\frac{r^2}{2}\left[\ln(r) - \frac{1}{2}\right]\right)\Bigg|_{R_2}^{R_1}, \tag{6.04aq}$$

then, finally,

$$Q = 2\pi\left(\frac{K(R_1^4 - R_2^4)}{16} + C_2\frac{(R_1^2 - R_2^2)}{2} + C_1\frac{R_1^2}{2}\left[\ln(R_1) - \frac{1}{2}\right] - C_1\frac{R_2^2}{2}\left[\ln(R_2) - \frac{1}{2}\right]\right). \tag{6.04ar}$$

Now, substituting C_1 and C_2,

$$Q = \frac{\pi K}{2}\left(\frac{R_1^4}{4} + \frac{R_2^4}{4} - \frac{R_1^2 R_2^2}{2} - (R_1^2 - R_2^2)\frac{R_1^2}{2} + \frac{(R_2^2 - R_1^2)^2}{4\ln(R_1/R_2)}\right) \tag{6.04as}$$

Now, re-expressing in terms of the requested variables,

$$C_2 = \frac{KR_1^2}{4}\left(-1 + \frac{1 - \Phi^2}{-\ln\Phi}\ln(R_1)\right), \qquad C_2 = \frac{KR_1^2}{4}\left(-1 + \frac{\Phi^2 - 1}{\ln\Phi}\ln(R_1)\right). \tag{6.04at}$$

And simplifying again,

$$\boxed{Q = \frac{\pi K R_1^4}{2}\left(\frac{(\Phi^2 - 1)^2}{4}\left(1 - \frac{1}{\ln\Phi}\right) + \frac{(\Phi^2 - 1)}{2}\right)} \tag{6.04au}$$

□

Problem Solution by MC, Fall 2008

Problem 6.04b

This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin

Solution:

- Now, factorizing taking into account that $R_1 = R_2 + H$, then $R_2 = R_1 - H$,

$$Q = \frac{K\pi}{8}\left(\frac{-2(R_1-H)^2R_1^2 + (R_1-H)^4\ln(\frac{R_1}{R_1-H}) + (R_1-H)^4 - R_1^4\ln(\frac{R_1}{R_1-H}) + R_1^4}{\ln(\frac{R_1}{R_1-H})}\right) \quad (6.04ba)$$

Now, let's substitute H using $F = H/R_1$,
$Q =$

$$\frac{K\pi}{8}\left(\frac{-2(R_1-FR_1)^2R_1^2 + (R_1-FR_1)^4\ln(\frac{R_1}{R_1-FR_1}) + (R_1-FR_1)^4 - R_1^4\ln(\frac{R_1}{R_1-FR_1}) + R_1^4}{\ln(\frac{R_1}{R_1-FR_1})}\right) \quad (6.04bb)$$

Now, taking the limit as $F \to 0$, but keeping the higher order terms,

$$Q = -\frac{2}{12}R_1^4 K\pi F^3, \quad (6.04bc)$$

and substituting the value of K, and the original variables,

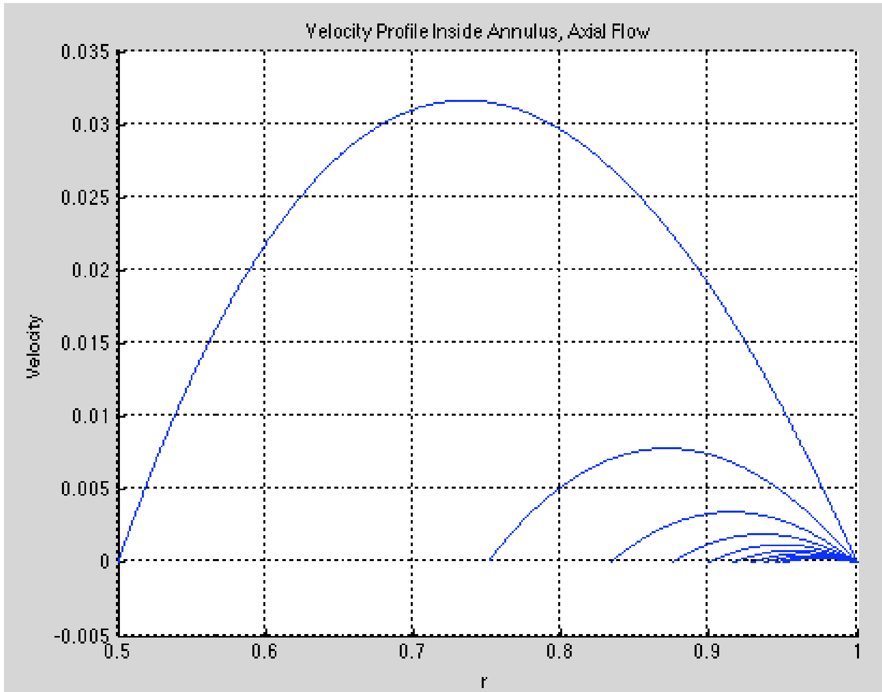
$$Q = -\frac{1}{6\mu}\frac{dP}{dx}R_1\pi H^3, \quad (6.04bd)$$

$$\boxed{Q = -\frac{H^3}{12\mu}\left(\frac{dP}{dx}\right)(2\pi R_1),} \quad (6.04be)$$

which corresponds to the solution of a pressure driven flow between two plates separated by a distance H, over a length equal to the average circumference of the annulus.

NOTE: SEE PLOTS OF THE SOLUTIONS USING THE ATTACHED MATLAB FILES, PLAY WITH THE SOLUTIONS TILL THE LIMITS MAKE SENSE TO YOU.

```
%This script plots the nondimensional velocity profiles for Shapiro 6.4, in
%case you ever have a doubt of the limits that you calculated, it is useful
%to plot the velocity profiles.
%
%Created by Marco Cartas for 2.25 Advanced Fluid Mechanics, MIT 2008
%
%      ////////////////////////// Outside Cylinder
%      ==========================                        XXXXXXXXXX
%  ^                            --->                    X/        \X
%  |                            -----> Flow            X/    __    \X
% b|   ==========================                     X/    /XX\    \X
%  | ^ //////////////////////////                     Xl   lXXXXl    lX
%  |a| ////////////////////////// Inner Cylinder      X\    \XX/    /X
%      ==========================                      X\          /X
%                               ---->                   X\________/X
%                               ---> Flow                XXXXXXXXXX
%      ==========================                      Circular Section
%      ////////////////////////// Outside Cylinder
%
%
%% Initial Variable Definition
a=0.5; %Initial Internal radius
b=1;   %Outside Radius
k=-1;  %Pressure Gradient
```

□

Problem Solution by MC, Fall 2008

Problem 6.04c

This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin

Solution:

- Now, taking the limit as$\Phi \rightarrow 0$ of part a) solution,

$$\lim_{\Phi\to 0} Q = \lim_{\Phi\to 0} \frac{\pi K R_1^4}{2}\left(\frac{(\Phi^2-1)^2}{4}\left(1-\frac{1}{\ln\Phi}\right)+\frac{(\Phi^2-1)}{2}\right), \tag{6.04ca}$$

$$\lim_{\Phi\to 0} Q = \lim_{\Phi\to 0} \frac{\pi K R_1^4}{2}\left(\frac{1}{4}(1)+\frac{-1}{2}\right), \tag{6.04cb}$$

$$\lim_{\Phi\to 0} Q = -\frac{\pi K R_1^4}{8}, \tag{6.04cc}$$

$$\lim_{\Phi\to 0} Q = -\frac{R_1^3}{16\mu}\left(\frac{dP}{dx}\right)(2\pi R_1), \tag{6.04cd}$$

which is the solution for Poiseuille flow for a simple tube. You may have guessed that the solution did not converge to this value, i.e. the velocity profile had a hole in the center, but this is wrong. The solution converges to the simple tube flow because as the inner cylinder becomes smaller, the area that it uses to transmit vorticity decreases, and as the area decreases, its influence decreases too (Think of a small string (hot wire) inside the tube for measuring flow, and think how small are the disturbances that it creates in the flow).

To further verify that the solution makes physical sense, let's look at the product $r * \tau_{viscous}$ to show that the viscous force per unit length decreases as $r \rightarrow 0$. Using the velocity profile, the viscous stress can be obtained,

$$\mu\frac{dv_x}{dr} = \mu K\left(2r - \frac{R_1^2 - R_2^2}{\ln(R_1/R_2)}\frac{1}{r}\right), \tag{6.04ce}$$

now, let's evaluate at $r = R_2$, and multiply by R_2,

$$\mu R_2\frac{dv_x}{dr}\bigg|_{R_2} = \mu K\left(2R_2^2 - \frac{R_1^2 - R_2^2}{\ln(R_1/R_2)}\right), \tag{6.04cf}$$

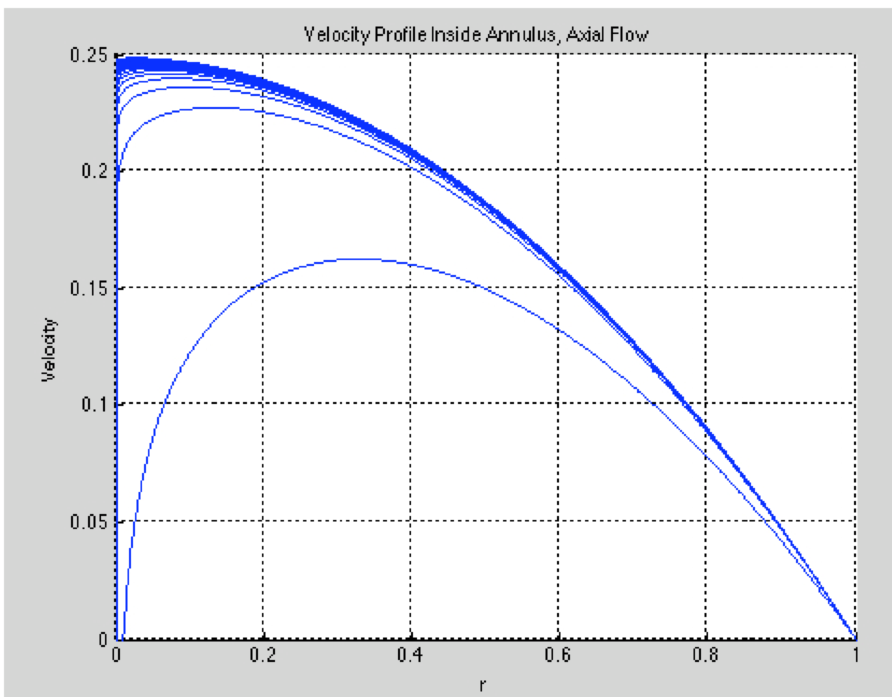
now, taking the limit as $R_2 \rightarrow 0$,

$$\lim_{R_2\to 0} \mu R_2\frac{dv_x}{dr}\bigg|_{R_2} = \lim_{R_2\to 0} \mu K(2R_2^2) = 0, \tag{6.04cg}$$

then, the net viscous force goes to 0 as the radius approaches 0.

NOTE: SEE PLOTS OF THE SOLUTIONS USING THE ATTACHED MATLAB FILES, PLAY WITH THE SOLUTIONS TILL THE LIMITS MAKE SENSE TO YOU.


```
%This script plots the nondimensional velocity profiles for Shapiro 6.4, in
%case you ever have a doubt of the limits that you calculated, it is useful
%to plot the velocity profiles.
%
%Created by Marco Cartas for 2.25 Advanced Fluid Mechanics, MIT 2008
%
%       ////////////////////////// Outside Cylinder
%       ==========================                       XXXXXXXXXX
%  ^                             --->                  X/        \X
%  |                             -----> Flow          X/    ___    \X
% b|   ==========================                    X/    /XX\    \X
%  | ^ //////////////////////////                    Xl   lXXXXl    lX
%  |a| ////////////////////////// Inner Cylinder     X\    \XX/    /X
%       ==========================                    X\          /X
%                                ---->                 X\________/X
%                                ---> Flow              XXXXXXXXXX
%       ==========================                    Circular Section
%       ////////////////////////// Outside Cylinder
%
%
%% Initial Variable Definition
a=0.1; %Initial Internal radius
b=1;   %Outside Radius
k=-1;  %Pressure Gradient
```



□

Problem Solution by MC, Fall 2008