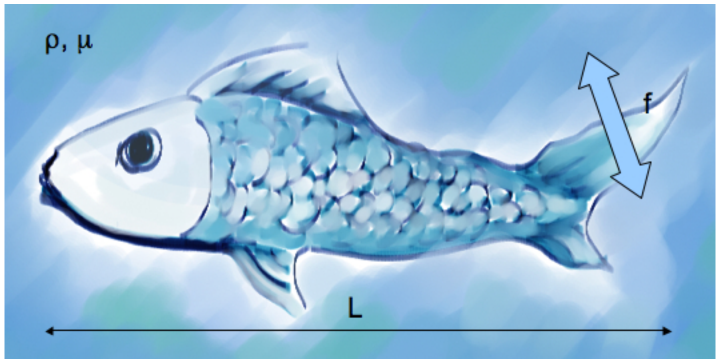
MIT Department of Mechanical Engineering
2.25 Advanced Fluid Mechanics

Problem 7.18
This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin



A researcher is concerned with the mechanics of fish propulsion. To determine how the thrust force generated by a fish of a given geometry depends on fish size (L = fish length) and on the frequency of oscillation of the tail (f, in cycles/sec), she builds a mechanical model, having this geometry, of length $L = 1\,\mathrm{m}$. She then mounts this model in a fixed position deep within a large tank containing stagnant water at room temperature, and measures the thrust force F, over a large range of frequency of tail oscillation. She finds that her data can be described by the empirical equation

$$F = \frac{0.49 \times 10^4 f^3}{1 + 0.74 \times 10^3 f} \text{ Newtons} \tag{7.18a}$$

where f is in cycles/sec.

(a) Suppose we want to infer, from these results, the thrust generated by fish of other sizes held in still water having different temperature (*i.e.* different density, viscosity). What relation must be satisfied between the frequency, size, and fluid condition of the real fish and of the model experiments

(b) From the empirical equation given above for the thrust of a 1 m model in room temperature water, develop a formula for the thrust of a fish of any given size and tail frequency, held in water at any given density and viscosity.

Solution:

We want an expression for the thrust force F in terms of all other variables, such that

$$F = f(L, \rho, \mu, f)$$

F	L	ρ	μ	f
[MLT^{-2}]	[L]	[ML^{-3}]	[ML^{-1}T^{-1}]	[T^{-1}]

Thus we have

$$\begin{aligned} n &= 5 \text{ variables} \\ k &= 3 \text{ primary variables} \\ \Rightarrow j &= 2 \text{ dimensionless groups} \end{aligned}$$

For our primary variables, we choose (1) a fluid property: ρ, (2) a flow parameter: f, and (3) a geometric parameter: L.

Follow the same procedure as in Problem 7.14(a) to find Π_1, and Π_2:

$$\begin{aligned} \Pi_1 &= \frac{F}{\rho L^4 f^2} \\ \Pi_2 &= \frac{\mu}{\rho L^2 f} = \frac{1}{\text{Re}} \qquad \text{Re} = \text{Reynolds number} = \frac{\text{Inertia}}{\text{Viscosity}} \\ \Rightarrow \frac{F}{\rho L^4 f^2} &= \phi\left(\frac{\mu}{\rho L^2 f}\right) \end{aligned}$$

(a) In order to ensure that the real fish and experiments are dynamically similar, the Reynolds number must remain constant

$$\boxed{\Rightarrow \quad \left(\frac{\mu}{\rho L^2 f}\right)_{\text{experiments}} = \left(\frac{\mu}{\rho L^2 f}\right)_{\text{real}}}$$

(b) We know that the thrust force can be expressed as:

$$\frac{F}{\rho L^4 f^2} = \phi\left(\frac{\mu}{\rho L^2 f}\right) = \phi\left(\frac{1}{\text{Re}}\right) \quad \Rightarrow \quad F = \phi\left(\frac{1}{\text{Re}}\right)\rho L^4 f^2 \tag{7.18b}$$

This says that the thrust force is some unkown function of the Reynolds number times f^2. However, the dependence of Eq. (7.18a) on f is a bit more complicated. Also, note that the coefficients in the given emipirical equation must have dimensions for the right-hand side to have the units of force:

$$F = \frac{\overbrace{(0.49 \times 10^4)}^{\text{units} = [\text{N}\cdot\text{s}^3]} f^3}{1 + \underbrace{(0.74 \times 10^3)}_{\text{units} = [\text{s}]}\ f}$$

Since Re $\sim f$, we will try to replicate Eq. (7.18a) with Eq (7.18b) by substituting Re (with an unknown coefficient) whenever we need an extra f:

$$F = \frac{C_1 \rho L^4 f^2 \text{Re}}{1 + C_2 \text{Re}} = \frac{C_1 (\rho^2 L^6/\mu) f^3}{1 + C_2(\rho L^2/\mu) f} \tag{7.18c}$$

For water at room temperature

$$\rho = 10^3 \, \frac{\text{kg}}{\text{m}^3} \qquad \mu = 10^{-3} \, \text{Pa} \cdot \text{s} = 10^{-3} \, \frac{\text{kg}}{\text{m} \cdot \text{s}}$$

Substituting these fluid properties and $L = 1\,\text{m}$, we see

$$F = \frac{C_1 (10^6 \, \frac{\text{kg}^2}{\text{m}^6})(1\,\text{m}^6)/(10^{-3} \, \frac{\text{kg}}{\text{m}\cdot\text{s}}) f^3}{1 + C_2 (10^3 \, \frac{\text{kg}}{\text{m}^3})(1\,\text{m}^2)/(10^{-3} \, \frac{\text{kg}}{\text{m}\cdot\text{s}}) f} = \frac{C_1 (10^9 \, \text{kg} \cdot \text{m} \cdot \text{s}) f^3}{1 + C_2 (10^6 \, \text{s}) f}$$

Comparing this to Eq. (7.18a), we see that

$$\left.\begin{array}{lll} C_1 \times 10^9 = 0.49 \times 10^4 & \Rightarrow & C_1 = 0.49 \times 10^{-5} \\ C_2 \times 10^6 = 0.74 \times 10^3 & \Rightarrow & C_1 = 0.74 \times 10^{-3} \end{array}\right\} \text{Note: these coefficients are dimensionless}$$

Now substitute the coefficients above into Eq. (7.18c) such that

$$F = \frac{(0.49 \times 10^{-5})(\rho L^2 f/\mu)}{1 + (0.74 \times 10^{-3})(\rho L^2 f/\mu)} \, \rho L^4 f^2$$

$$\Rightarrow \boxed{\frac{F}{\rho L^2 f/\mu} = \Pi_1 = \frac{(0.49 \times 10^{-5}) \, \text{Re}}{1 + (0.74 \times 10^{-3}) \, \text{Re}}}$$

Recall, that the coefficients in front of Re are dimensionless such that both sides of the above equation are dimensionless. This new equation is more general than the empirical equation given because changes in ρ, μ, and L are taken into account (the original equation was only valid for $\rho = 10^3 \, \frac{\text{kg}}{\text{m}^3}$, $\mu = 10^{-3}\,\text{Pa} \cdot \text{s}$, $L = 1\,\text{m}$).

□

Problem Solution by Tony Yu/ MC(Updated), Fall 2008

2.25 Advanced Fluid Mechanics
Fall 2013