MIT Department of Mechanical Engineering
2.25 Advanced Fluid Mechanics

Problem 7.12

This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin

Consider an incompressible flow through a series of geometrically similar machines such as fans, pumps, hydraulic turbines, etc. If Q denotes volume flow, ω rotational speed, D impeller diameter, μ fluid viscosity, and ρ fluid density,

(a) show that dynamic similarity requires that $Q/\omega D^3$ and $\rho Q/\mu D$ be fixed.

(b) Show that if $Q/\omega D^3$ and $\rho Q/\mu D$ are fixed in a series of tests, then $\Delta P/\rho\omega^2 D^2$ must remain constant, where ΔP is the change in head across the machine, expressed in pressure units.

(c) Find the form of the relation between the work output per unit mass of fluid W, and the the given variables, in a series of tests where $Q/\omega D^3$ and $\rho Q/\mu D$ are fixed.

Solution:

(a) The variables in the problem are related through $f(Q, \omega, D, \mu, \rho, \Delta P) = 0$ where

Q [L^3T^{-1}] : Volume flow rate
ω [T^{-1}] : Rotational Speed
D [L] : Impeller diameter
μ [$ML^{-1}T^{-1}$] : Fluid viscosity
ρ [ML^{-3}] : Fluid density
ΔP [$ML^{-1}T^{-1}$] : Change in head across machine

As our primary variables, we pick ρ for the fluid, ω for the flow and D for the geometry. We have

$$n = 6 \text{ variables}$$
$$r = 3 \text{ primary dimensions}$$
$$\Rightarrow j = 6 - 3 = 3 \text{ dimensionless groups}$$

Now,

$$\Pi_1 = \frac{Q}{\rho^a \omega^b D^c} \tag{7.12a}$$

By inspection, we find $a = 0$, $b = 1$ and $c = 3$. Therefore

$$\Pi_1 = \frac{Q}{\omega D^3} \tag{7.12b}$$

Similarly, we find

$$\Pi_2 = \frac{\mu}{\rho \omega D^2} \tag{7.12c}$$

Let

$$\Pi' = \frac{\Pi_1}{\Pi_2} \tag{7.12d}$$
$$= \frac{Q}{\omega D^3} \times \frac{\rho \omega D^2}{\mu} \tag{7.12e}$$
$$\Rightarrow \Pi' = \frac{\rho Q}{\mu D} \tag{7.12f}$$

Therefore, dynamic similarity requires that

$$\Pi_1 = \frac{Q}{\omega D^3} = C_1 \tag{7.12g}$$
$$\text{and} \quad \Pi' = \frac{\rho Q}{\mu D} = C_2 \tag{7.12h}$$

where C_1 and C_2 are constants.

(b) The third non-dimensional group is given by

$$\Pi_3 = \frac{\Delta P}{\rho \omega^2 D^2} \tag{7.12i}$$

Therefore, from the Buckingham Π theorem,

$$\Pi_3 = f(\Pi_1, \Pi') \tag{7.12j}$$

$$\Rightarrow \frac{\Delta P}{\rho\omega^2 D^2} = f(\frac{Q}{\omega D^3}, \frac{\rho Q}{\mu D}) \tag{7.12k}$$

Now if $\Pi_1 = \frac{Q}{\omega D^3}$ and $\Pi' = \frac{\rho Q}{\mu D}$ are constants, then $f(\Pi_1, \Pi') = f(C_1, C_2) = C$ where C is a constant. Hence, equation (7.12k) implies

$$\boxed{\frac{\Delta P}{\rho\omega^2 D^2} = C} \tag{7.12l}$$

(c) The work output w is given by

$$w = Q\Delta P \tag{7.12m}$$

We know from equation (7.12h) that $Q = C_1\omega D^3$ and from equation 7.12l that $\Delta P = C\rho\omega^2 D^2$. Substituting this into equation (7.12m), we have

$$w = C_1 C\rho\omega^3 D^5 = K\rho\omega^3 D^5 \tag{7.12n}$$

where K is a constant. Thus we have per unit mass that

$$\boxed{W = \frac{w}{\rho D^3} = KD^2\omega^3} \tag{7.12o}$$

□

Problem Solution by Aditya Jaishankar, Fall 2011