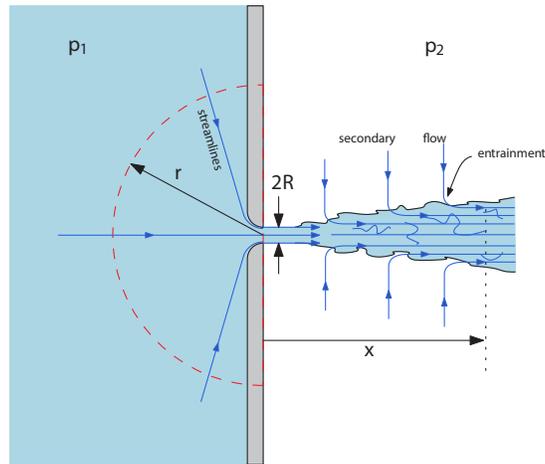


MIT Department of Mechanical Engineering
2.25 Advanced Fluid Mechanics

Problem 5.13

This problem is from “Advanced Fluid Mechanics Problems” by A.H. Shapiro and A.A. Sonin



Two large compartments are separated by a wall in which there is a small nozzle, or hole, of radius R . The pressure p_1 in the left-hand compartment far from the nozzle is greater than the pressure p_2 in the right-hand compartment, and a steady volume flow Q takes place from (1) to (2).

The flow through the nozzle is an incompressible, high Reynolds number flow, typical of the ones rather loosely termed “inviscid”. However, as in all such flows, viscous forces are responsible for the phenomenon of flow separation which gives rise to a profound difference between the inflow and outflow regions of the nozzle flow field.

In compartment (1), on the inflow side, the flow is directed approximately radially inward toward the nozzle entrance until one gets close to the nozzle, and is essentially inviscid.

In compartment (2), however, the flow separates from the boundaries and emerges from the nozzle as a horizontal jet.

Inside compartment (2) viscous forces slow the jet and also cause it to become turbulent and to mix with the surrounding fluid, some of which is dragged along with the jet as the latter penetrates into the compartment and gradually slows down. The process whereby the jet drags some of the ambient fluid along with it is called “entrainment”. and gives rise in compartment (2) to a secondary bulk flow which is directed approximately radially inward toward the jet as sketched. The velocities associated with this secondary flow are relatively small, however, and the pressure in compartment (2) can be modeled as being approximately uniform.

- (a) Consider a disc-shaped portion of the wall extending a radial distance r from the nozzle centerline. Using a control volume whose left side is a hemisphere of radius r , show that as $r/R \rightarrow \infty$, the x -component of external force required to hold this portion of the wall in place is given by

$$F = -(p_1 - p_2)\pi r^2 + \rho \frac{Q^2}{\pi R^2}$$

- (b) Consider the integrals of (i) mass flux and (ii) x -direction momentum flux across a plane at station x to the right of the nozzle exit. Do these integrals grow, decrease, or remain constant as x increases? How do they compare with their values at the nozzle exit plane?

Gravity is to be neglected in this problem.

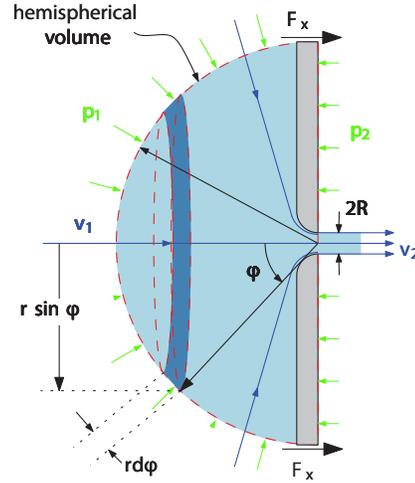
Solution:

Apply Bernoulli from compartment (1) to (2)

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$

- (a) If r is much larger than R , then we can say that $v_2 \gg v_1$, such that

$$v_2^2 = \frac{2(p_1 - p_2)}{\rho} \quad (5.13a)$$



As the fluid moves from First we make a table of the relevant parameters

	$\hat{\mathbf{n}}$	\mathbf{v}	\mathbf{v}_c	$(\mathbf{v} \cdot \hat{\mathbf{e}}_x) = v_x$	Pressure
(1)	$\hat{\mathbf{e}}_r$	$-v_r \hat{\mathbf{e}}_r$	0	$v_r \cos \phi$	p_1
(2)	jet	$v_2 \hat{\mathbf{e}}_x$	0	v_2	p_2
	wall	$\hat{\mathbf{e}}_x$	0	0	p_2

Apply conservation of momentum

$$\begin{aligned} \sum \mathbf{F} \cdot \hat{\mathbf{e}}_x &= \left[\frac{d}{dt} \int_{CV} \rho \mathbf{v} dV + \int_{CS} \rho \mathbf{v} (\mathbf{v} - \mathbf{v}_c) \cdot \hat{\mathbf{n}} dA \right] \cdot \hat{\mathbf{e}}_x \\ [p_1 \pi r^2 - p_2 \pi r^2] + F_x &= \int_{(1)} \rho \underbrace{(-v_r)}_{\mathbf{v} \cdot \hat{\mathbf{n}}} \underbrace{(v_r \cos \phi)}_{v_x} \underbrace{2\pi r^2 \sin \phi d\phi}_{dA} + \underbrace{\rho v_2^2 \pi R^2}_{(2)} \\ F_x + (p_1 - p_2) \pi r^2 &= -\rho v_r^2 2\pi r^2 \int_0^{\pi/2} \cos \phi \sin \phi d\phi + \rho v_2^2 \pi R^2 \\ &= -\rho v_r^2 2\pi r^2 + \rho v_2^2 \pi R^2 \end{aligned}$$

To find v_r , apply mass conservation

$$\begin{aligned} \frac{d}{dt} \int_{CV} \rho dV + \int_{CS} \rho (\mathbf{v} - \mathbf{v}_c) \cdot \hat{\mathbf{n}} dA &= 0 \\ \int_0^{\pi/2} \rho (-v_r) 2\pi r^2 \sin \phi d\phi + \rho v_2 \pi R^2 &= 0 \\ -\rho v_r r^2 2\pi + \rho v_2 \pi R^2 = 0 &\Rightarrow \boxed{v_r = \frac{v_2 R^2}{2r^2}} \end{aligned}$$

Finally, substitute the above relation for v_r into conservation of momentum

$$\begin{aligned}
 F_x &= -(p_1 - p_2)\pi r^2 - \rho \left(\frac{v_2 R^2}{2r^2} \right)^2 \pi r^2 + \rho v_2^2 \pi R^2 \\
 &= -(p_1 - p_2)\pi r^2 + \rho v_2^2 \pi R^2 \left(1 - \frac{R^2}{4r^2} \right)
 \end{aligned}
 \tag{5.13b}$$

As $r/R \rightarrow \infty$

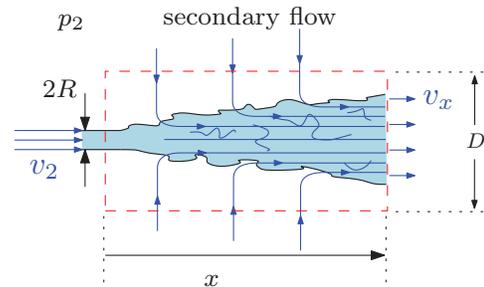
$$F_x = -(p_1 - p_2)\pi r^2 + \rho v_2^2 \pi R^2 = -(p_1 - p_2)\pi r^2 + \rho \frac{Q^2}{\pi R^2}$$

If we substitute our previous relation for v_2 [Eq. (5.13a)] into Eq. (5.13b)

$$F_x = (p_1 - p_2)\pi r^2 \left[-1 + 2 \left(\frac{R}{r} \right)^2 - \frac{1}{2} \left(\frac{R}{r} \right)^4 \right]$$

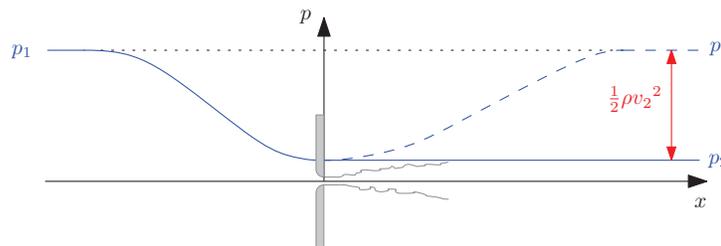
Note that we have applied conservation of momentum on the fluid, and as a result, this is the force of the wall *on the fluid*. The force of the fluid on the wall is equal and opposite F_x .

(b) Consider a control volume of length x :



As x increases, the secondary flow into this control volume increases and thus, the mass flux through the control volume increases. However, since the x -component of the secondary flow velocity is small (if we choose the diameter of our control volume D to be large compared to diameter of the flow), the x -momentum flux is relatively constant with increasing x . Although the x -momentum is constant, it does diffuse radially and we must take a control volume with larger D as we increase x .

Far away from the wall ($x \rightarrow \infty$) we expect the fluid velocity to go to zero. If we applied Bernoulli's equation from (1) a point far to the left of the wall to (2) a point far to the right of the wall, we would expect the pressures to be equal since the velocities are roughly zero.



However, we cannot apply Bernoulli between these two points, because viscosity is important far from the wall and is the reason that fluid is entrained. Far to the right of the wall, the pressure does not increase to p_1 and energy is “burned” by viscous dissipation. The fluid will flow smoothly go down a pressure gradient (favorable pressure gradient), but not up a pressure gradient (adverse pressure gradient).

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Fall 2013

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