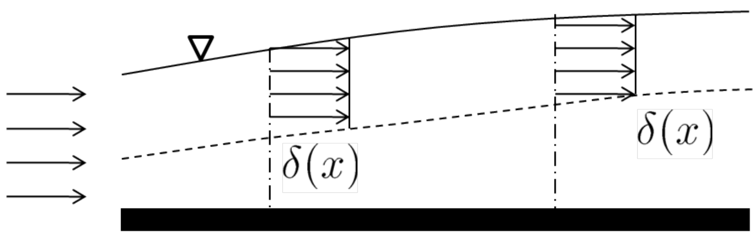
Massachusetts Institute of Technology
Department of Mechanical Engineering
2.25 Advanced Fluid Mechanics

2009 Final Exam Solution

Question 1



Solution:

The given horizontal velocity profile $u(x,y)$ is given inside the boundary layer as

$$\frac{u(x,y)}{U(x)} = a(x)\left(\frac{y}{\delta(x)}\right) + b(x)\left(\frac{y}{\delta(x)}\right)^3, \qquad 0 < y < \delta(x)$$

a) Determine $a(x)$ and $b(x)$

Firstly, substitute $y = \delta(x)$ into the given velocity profile, in which $u(x,\delta(x)) = U(x)$.

$$\Rightarrow \qquad 1 = a(x) + b(x) \tag{1}$$

And apply shear-free boundary condition at $y = \delta(x)$, i.e.,

$$\mu\frac{\partial u}{\partial y} = 0 \quad \Rightarrow \quad a(x) + 3b(x) = 0 \tag{2}$$

Therefore, both $a(x)$ and $b(x)$ are constant.

$$\boxed{a(x) = \frac{3}{2}}, \qquad \boxed{b(x) = -\frac{1}{2}} \tag{3}$$

b) Expression for $U(L)$

Since inviscid assumption is valid in outer flow, apply the Bernoulli from $x = 0$ to $x = L$ along the top streamline.

$$p_a + \frac{1}{2}\rho U_\infty^2 + \rho g d = p_a + \frac{1}{2}\rho U(L)^2 + \rho g \delta(L) \tag{4}$$

$$\Rightarrow \boxed{U(L) = \left(U_\infty^2 + 2g(d - \delta(L))\right)^{1/2}} \tag{5}$$

c) Determine $\delta(L)$

Apply mass conservation between $x = 0$ and $x = L$.

$$U_\infty \cdot d = \int_0^{\delta(L)} u(L, y)\, dy \tag{6}$$

$$= \int_0^{\delta(L)} U(L) \left(\frac{3}{2}\left(\frac{y}{\delta(x)}\right) - \frac{1}{2}\left(\frac{y}{\delta(x)}\right)^3 \right) dy \tag{7}$$

$$= \cdots = \frac{5}{8} U(L)\delta(L) \tag{8}$$

$$\Rightarrow \quad \delta(L) = \frac{8U_\infty d}{5U(L)} \tag{9}$$

Using $U(L)$ which was obtained in part b),

$$\delta(L) = \frac{8U_\infty d}{5\left(U_\infty^2 + 2g(d - \delta(L))\right)^{1/2}} \tag{10}$$

d) Pressure distribution in vertical direction at $x = L$

Navier-Stokes equation in y-direction is

$$0 = -\frac{\partial p}{\partial y} - \rho g \qquad \Leftrightarrow \qquad \frac{\partial p}{\partial y} = -\rho g \tag{11}$$

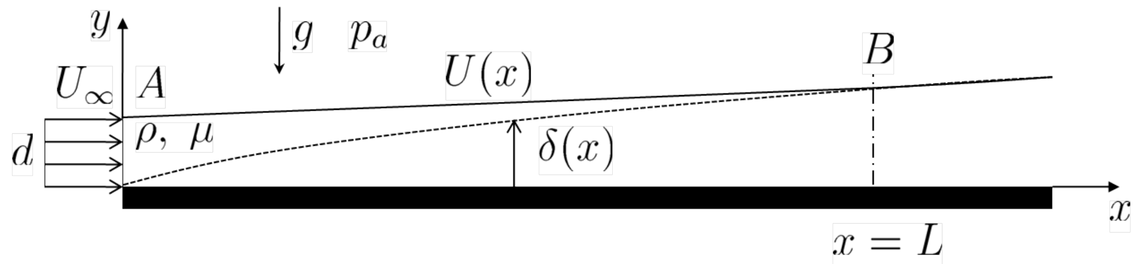
The boundary condition for this ODE is the atmospheric pressure at $y = \delta(L)$. Then,

$$p(L, y) = p_a + \rho g\left(\delta(L) - y\right) \tag{12}$$

This is nothing but the hydrostatic force balance.

e) Horizontal force acting on the plate

Apply the momentum conservation principle for the control volume



Momentum flows in the x-direction through planes A and B and through the $y = 0$ plane via shear stresses acting on the plate. No shear stresses contribute to the x-component of the momentum flux on plane B. No momentum is flowing through the free surface.

So the drag on the plate of the length L is equal to the momentum flux deficit in x-direction :

$$D = \rho U^2 d + \rho g \int_0^d (d - y)\, dy - \rho \int_0^{\delta(L)} u(L, y)^2\, dy - \rho g \int_0^{\delta(L)} (\delta - y)\, dy \quad (13)$$

$$= \rho U^2 d + \rho g \left(\frac{d^2}{2} - \frac{\delta(L)^2}{2} \right) - \rho U(L)^2 \delta(L) \times \frac{17}{35} \quad (14)$$

f) Shear stress for $x > L$

The velocity profile for region $x > L$ is following the given function. Shear stress becomes

$$\tau_{xy} = \mu \frac{\partial u}{\partial y}|_{y=0} = \frac{2}{3} \frac{U(x)}{\delta(x)} \quad (15)$$

We do not know how $U(x)$ and $\delta(x)$ evolve along the distance. To obtain this, use mass conservation.

$$Q = U_\infty d = \frac{5}{8} U(L) \delta(L) = \frac{5}{8} U(x) \delta(x) \quad (16)$$

$$\Rightarrow U(x) = \frac{8}{5} \frac{Q}{\delta(x)} \quad (17)$$

Plug this into shear stress.

$$\tau_{xy} = \frac{2}{3} \times \frac{8}{5} \times \frac{Q}{\delta(x)^2} = \frac{16}{15} \frac{Q}{\delta(x)^2} \quad (18)$$

$$or \qquad \tau_{xy} = \frac{2}{3} \times \frac{5}{8} \times \frac{U(x)^2}{Q} = \frac{5}{12} \frac{U(x)^2}{Q} \quad (19)$$

Assume that over a very thin layer on the free surface at the edge of the boundary layer the flow is inviscid. Then Bernoulli applies and

$$\frac{1}{2}\rho U_\infty^2 + \rho g d = \frac{1}{2}\rho U(x)^2 + \rho g \delta(x) \tag{20}$$

Combining with Equation (17) gives

$$\frac{1}{2}\rho U_\infty^2 + \rho g d = \frac{1}{2}\rho U(x)^2 + \frac{8\rho g Q}{5U(x)} \tag{21}$$

This is a cubic equation for $U(x)$ in terms of known quantities. Thus, the shear stress τ_{xy} can be obtained by using the solution of the above equation.

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Problem Solution by J.Kim

