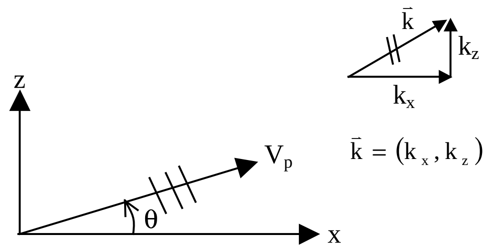
2.20 - Marine Hydrodynamics
Lecture 21

6.4 Superposition of Linear Plane Progressive Waves

1. **Oblique Plane Waves**



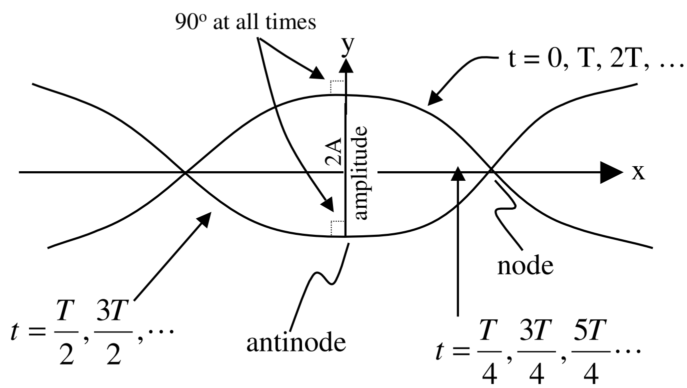
(Looking up the y-axis from below the surface)

Consider wave propagation at an angle θ to the x-axis

$$\begin{aligned}
\eta &= A\cos(\overbrace{kx\cos\theta + kz\sin\theta}^{\vec{k}\cdot\vec{x}} - \omega t) = A\cos(k_x x + k_z z - \omega t) \\
\phi &= \frac{gA}{\omega}\frac{\cosh k(y+h)}{\cosh kh}\sin(kx\cos\theta + kz\sin\theta - \omega t) \\
\omega &= gk\tanh kh; k_x = k\cos\theta, k_z = k\sin\theta, k = \sqrt{k_x + k_z}
\end{aligned}$$

2. **Standing Waves**

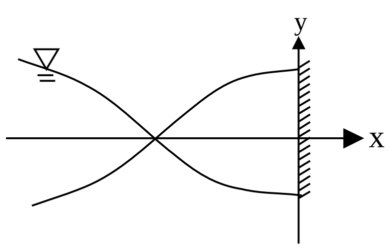
$$\eta = A\cos(kx-\omega t) + A\cos(-kx-\omega t) = 2A\cos kx\cos\omega t$$
$$\phi = -\frac{2gA}{\omega}\frac{\cosh k(y+h)}{\cosh kh}\cos kx\sin\omega t$$



$$\frac{\partial\eta}{\partial x} \sim \frac{\partial\phi}{\partial x} = \cdots\sin kx = 0 \text{ at } x = 0, \frac{n\pi}{k} = \frac{n\lambda}{2}$$

Therefore, $\left.\frac{\partial\phi}{\partial x}\right|_x = 0$. To obtain a standing wave, it is necessary to have perfect reflection at the wall at $x = 0$.

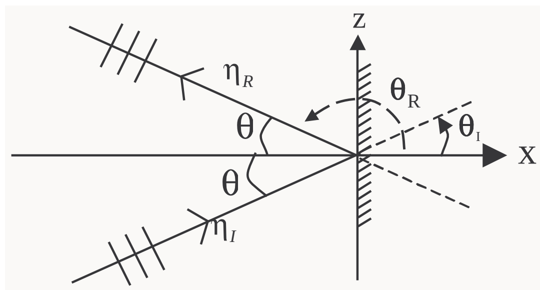
Define the reflection coefficient as $R \equiv \frac{A_R}{A_I}\ (\leq 1)$.



$$A_I = A_R$$
$$R = \frac{A_R}{A_I} = 1$$

3. **Oblique Standing Waves**

$$\begin{aligned}\eta_I =& A\cos\left(kx\cos\theta + kz\sin\theta - \omega t\right)\\ \eta_R =& A\cos\left(kx\cos\left(\pi-\theta\right) + kz\sin\left(\pi-\theta\right) - \omega t\right)\end{aligned}$$



$$\theta_R = \pi - \theta_I$$

Note: same A, R = 1.

$$\eta_T = \eta_I + \eta_R = 2A\underbrace{\cos\overbrace{\left(kx\cos\theta\right)}^{k_x x}}_{\text{standing wave in x}}\underbrace{\cos\overbrace{\left(kz\sin\theta-\omega t\right)}^{k_z z-\omega t}}_{\text{propagating wave in z}}$$

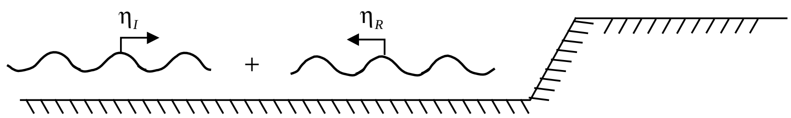
and

$$\lambda_x = \frac{2\pi}{k\cos\theta}; \qquad V_{P_x} = 0; \qquad \lambda_z = \frac{2\pi}{k\sin\theta}; \qquad V_{P_z} = \frac{\omega}{k\sin\theta}$$

Check:

$$\frac{\partial\phi}{\partial \mathrm{x}} \sim \frac{\partial\eta}{\partial \mathrm{x}} \sim \cdots\sin\left(kx\cos\theta\right) = 0 \text{ on } x = 0$$

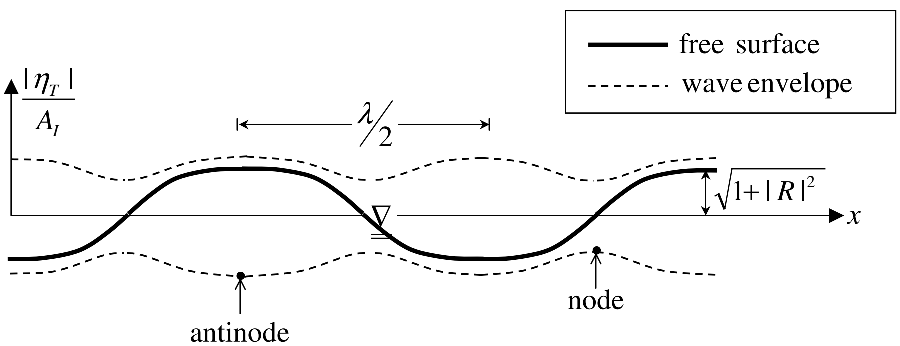
4. **Partial Reflection**



$$\eta_I = A_I \cos(kx - \omega t) = A_I Re\left\{e^{i\ kx - \omega t}\ \right\}$$
$$\eta_R = A_R \cos(kx + \omega t + \delta) = A_I Re\left\{\mathrm{R}\ e^{-i\ kx\ \ \omega t}\ \right\}$$

R: Complex reflection coefficient

$$\mathrm{R}\ = \ |\mathrm{R}|\, e^{-i\delta}, |\mathrm{R}| = \frac{A_R}{A_I}$$
$$\eta_T = \eta_I + \eta_R = A_I Re\left\{e^{i\ kx - \omega t}\ \left(1 + \mathrm{R}e^{-\ ikx}\right)\right\}$$
$$|\eta_T|\ \ = A_I\left[1 + |\mathrm{R}|\ \ + 2\,|\mathrm{R}|\cos(2kx + \delta)\right]$$



At node,

$$|\eta_T| = |\eta_T|\ \ \ \ = A_I(1 - |R|)\ \ at \cos(2kx + \delta) = -1 \text{ or } 2kx + \delta = (2n+1)\pi$$

At antinode,

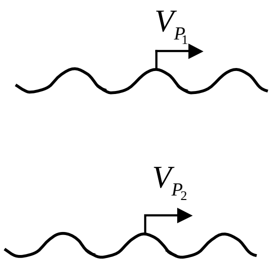
$$|\eta_T| = |\eta_T|\ \ \ \ = A_I(1 + |R|)\ \text{ at }\ \cos(2kx + \delta) = 1 \text{ or } 2kx + \delta = 2n\pi$$

$$2kL = 2\pi \text{ so } L = \frac{\lambda}{2}$$

$$|R| = \frac{|\eta_T|\ \ \ \ - |\eta_T|}{|\eta_T|\ \ \ \ + |\eta_T|} = |R(k)|$$

5. **Wave Group**

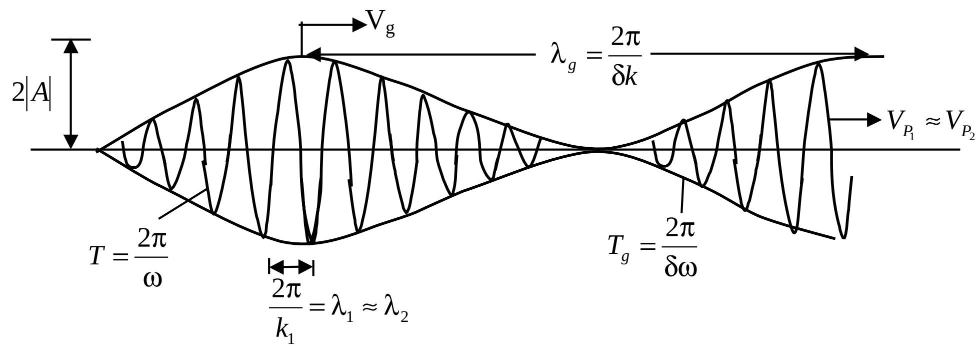
2 waves, same amplitude A and direction, but ω and k very close to each other.



$$\eta \;\; = \Re\left(Ae^{i\;k_1 x-\omega_1 t}\;\right)$$
$$\eta \;\; = \Re\left(Ae^{i\;k_2 x-\omega_2 t}\;\right)$$

$$\omega\;,\;\; = \omega\;,\;\; (k\;,\;)\;\text{ and } V_{P_1} \approx V_{P_2}$$

$$\eta_T = \eta \;\; + \eta \;\; = \Re\left\{Ae^{i\;k_1 x-\omega_1 t}\;\left[1+e^{i\;\delta k x-\delta\omega t}\;\right]\right\}\;\text{ with } \delta k = k \;\; - k \;\;\text{ and } \delta\omega = \omega \;\; - \omega$$



$$\left.\begin{aligned} |\eta_T| \quad &= 2\,|A|\;\text{ when } \delta k x - \delta\omega t = 2n\pi \\ |\eta_T| \quad &= 0 \text{ when } \delta k x - \delta\omega t = (2n+1)\,\pi \end{aligned}\right\}\; x_g = V_g t,\; \delta k V_g t - (\delta\omega)\,t = 0 \text{ then } V_g = \frac{\delta\omega}{\delta k}$$

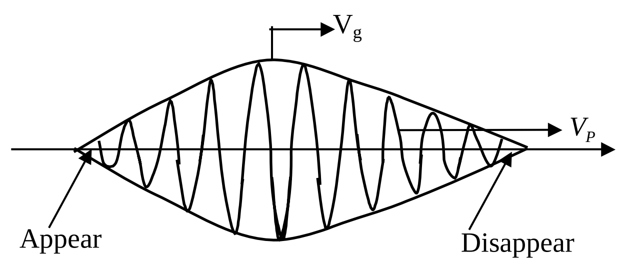
In the limit,

$$\delta k, \delta\omega \to 0, V_g = \left.\frac{d\omega}{dk}\right|_{k_1 \approx k_2 \approx k},$$

and since

$$\begin{aligned} \omega &= gk \tanh kh \Rightarrow \\ V_g &= \underbrace{\left(\frac{\omega}{k}\right)}_{V_p} \underbrace{\frac{1}{2}\left(1 + \frac{2kh}{\sinh 2kh}\right)}_{n} \end{aligned}$$

$$\left.\begin{array}{l} \text{(a) deep water } kh >> 1 \\ n = \frac{V_g}{V_p} = \text{-} \\ \text{(b) shallow water } kh << 1 \\ n = \frac{V_g}{V_p} = 1 \text{ (no dispersion)} \\ \text{(c) intermediate depth} \\ \text{-} < n < 1 \end{array}\right\} V_g \leq V_p$$



6.5 Wave Energy - Energy Associated with Wave Motion.

For a single plane progressive wave:

Energy per unit surface area of wave	
• Potential energy PE	• Kinetic energy KE
PE without wave $= \int_{-h} \rho g y dy = -\text{-}\rho g h$	$\text{KE}_{wave} = \int_{-h}^{\eta} dy \text{-} \rho (u \;\; + v \;\;)$
PE with wave $\int_{-h}^{\eta} \rho g y dy = \text{-}\rho g (\eta \;\; - h \;\;)$	Deep water $= \cdots = \underbrace{\text{-}\rho g A}_{\text{KE const in x,t}}$ to leading order
$\text{PE}_{wave} = \text{-}\rho g \eta \;\; = \text{-}\rho g A \;\; \cos \;\; (kx - \omega t)$	Finite depth $= \cdots$
Average energy over one period or one wavelength	
$\overline{\text{PE}}_{\text{wave}} = \text{-}\rho g A$	$\overline{\text{KE}}_{\text{wave}} = \text{-}\rho g A$ at **any** h

- Total wave energy in **deep** water:

  $\text{E} = \text{PE} + \text{KE} = \text{-}\rho g A \;\; \left[\cos \;\; (kx - \omega t) + \text{-}\right]$

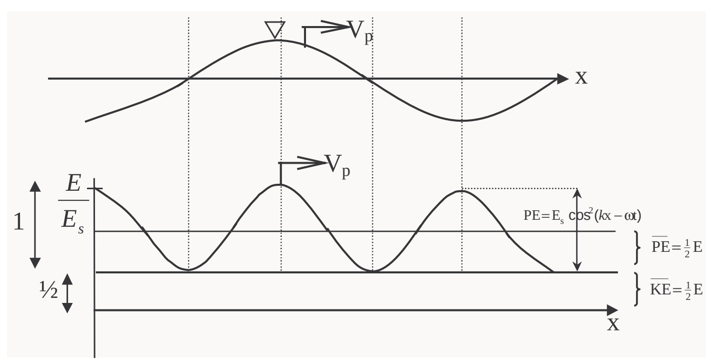
- Average wave energy E (over 1 period or 1 wavelength) for **any** water depth:

  $\overline{\text{E}} = \text{-}\rho g A \;\; [\underset{\substack{\uparrow \\ PE}}{—} + \underset{\substack{\uparrow \\ KE}}{—}] = \text{-}\rho g A \;\; = E_s,$

  $E_s \equiv$ Specific Energy: total average wave energy per unit surface area.

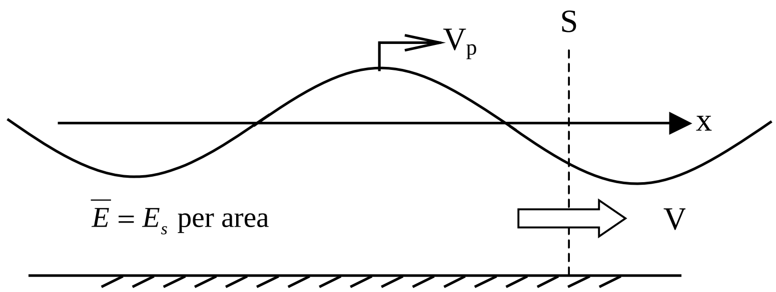
- Linear waves: $\overline{\text{PE}} = \overline{\text{KE}} = \frac{1}{2} E_s$ (equipartition).

- Nonlinear waves: $\overline{\text{KE}} > \overline{\text{PE}}$.



Recall: $\cos \;\; x = \text{-} + \text{-} \cos 2x$

6.6 Energy Propagation - Group Velocity



Consider a fixed control volume V to the right of 'screen' S. Conservation of energy:

$$\underbrace{\frac{dW}{dt}}_{\text{rate of work done on S}} = \underbrace{\frac{dE}{dt}}_{\text{rate of change of energy in V}} = \underbrace{\mathfrak{J}}_{\text{energy flux left to right}}$$

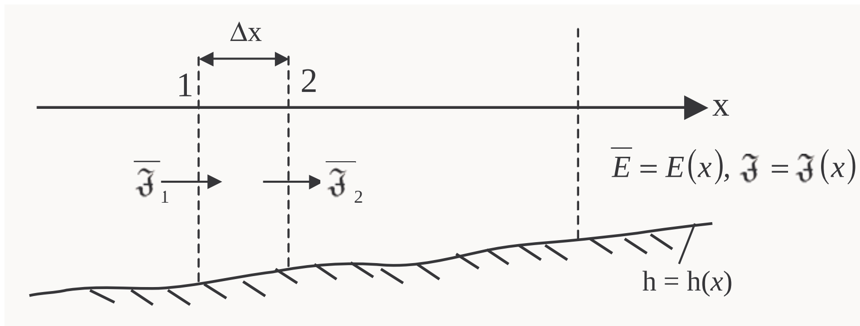
where

$$\mathfrak{J} = \int_{-h}^{\eta} pu\ dy \text{ with } p = -\rho\left(\frac{d\phi}{dt} + gy\right) \text{ and } u = \frac{\partial\phi}{\partial x}$$

$$\overline{\mathfrak{J}} = \underbrace{\left(\text{-}\rho g A\ \right)}_{\overline{E}} \underbrace{\underbrace{\left(\frac{\omega}{k}\right)}_{V_p} \underbrace{\left[\text{-}\left(1 + \frac{kh}{kh}\right)\right]}_{n}}_{V_g} = \overline{E}\left(nV_p\right) = \overline{E}V_g$$

e.g. A = 3m, T = 10 sec $\rightarrow \overline{\mathfrak{J}} = 400KW/m$

6.7 Equation of Energy Conservation



$$\left(\overline{\mathfrak{I}} \;\; - \overline{\mathfrak{I}} \;\;\right) \Delta t = \Delta \overline{E} \Delta x$$

$$\overline{\mathfrak{I}} \;\; = \overline{\mathfrak{I}} \;\; + \left.\frac{\partial \overline{\mathfrak{I}}}{\partial x}\right| \;\; \Delta x + \cdots$$

$$\frac{\partial \overline{E}}{\partial t} + \frac{\partial \overline{\mathfrak{I}}}{\partial x} = 0, \text{ but } \overline{\mathfrak{I}} = V_g \overline{E}$$

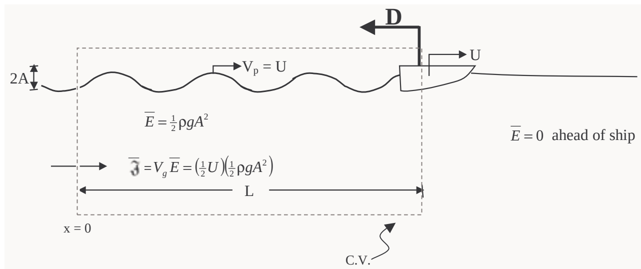
$$\frac{\partial \overline{E}}{\partial t} + \frac{\partial}{\partial x}\left(V_g \overline{E}\right) = 0$$

1. $\dfrac{\partial \overline{E}}{\partial t} = 0, V_g \overline{E} =$ constant in x for any $h(x)$.
2. $V_g =$ constant (i.e., constant depth, $\delta k << k$)

$$\left(\frac{\partial}{\partial t} + V_g \frac{\partial}{\partial x}\right) \overline{E} = 0, \text{ so } \overline{E} = \overline{E}\left(x - V_g t\right) \text{ or } A = A\left(x - V_g t\right)$$

i.e., wave packet moves at V_g.

6.8 Steady Ship Waves, Wave Resistance



- *Ship wave resistance drag* D_w

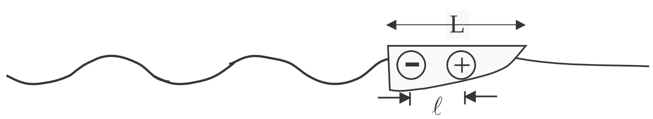
  Rate of work done = rate of energy increase

$$D_w U + \overline{\mathfrak{J}} = \frac{d}{dt}\left(\overline{E}L\right) = \overline{E}U$$

$$\underset{\text{force / length}}{D_w} = \frac{1}{U}(\overline{E}U - \overbrace{\overline{E}U/2}^{\text{deep water}}) = -\overline{E} = \underset{\text{energy / area}}{-\rho g A} \Rightarrow D_w \propto A$$

- *Amplitude of generated waves*

  The amplitude A depends on U and the ship geometry. Let $\ell \equiv$ effective length.



To *approximate* the wave amplitude A superimpose a bow wave (η_b) and a stern wave (η_s).

$$\begin{aligned}
\eta_b &= a\cos(kx) \text{ and } \eta_s = -a\cos(k(x+\ell)) \\
\eta_T &= \eta_b + \eta_s \\
A &= |\eta_T| \quad = 2a\left|\sin\left(\text{-}k\ell\right)\right| \leftarrow \text{ envelope amplitude} \\
D_w &= \text{-}\rho g A \ = \rho g a \ \sin\ \left(\text{-}k\ell\right) \Rightarrow D_w = \rho g a \ \sin\ \left(\text{-}\tfrac{g\ell}{U^2}\right)
\end{aligned}$$

- *Wavelength of generated waves* To obtain the wave length, observe that the *phase speed* of the waves must equal U. For deep water, we therefore have

$$V_p = U \Rightarrow \frac{\omega}{k} = U \underset{\text{water}}{\overset{\text{deep}}{\longrightarrow}} \sqrt{\frac{g}{k}} = U, \text{ or } \lambda = 2\pi\frac{U}{g}$$

- *Summary* Steady ship waves in deep water.

$$U = \text{ship speed}$$

$$V_p = \sqrt{\frac{g}{k}} = U; \text{ so } k = \frac{g}{U} \text{ and } \lambda = 2\pi\frac{U}{g}$$

$$L = \text{ship length}, \ \ell \sim L$$

$$D_w = \rho g a \ \sin \ \left(-\tfrac{g\ell}{U^2}\right) \cong \rho g a \ \sin \ \left(\frac{1}{2F_{r_L}}\right) \cong \rho g a \ \sin \ \left(\frac{1}{2F_{r_L}}\right)$$