2.161 Signal Processing: Continuous and Discrete
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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING
2.161 Signal Processing - Continuous and Discrete

Parallel Derivations of the Z and Laplace Transforms [1]

The following is a summary of the derivation of the Laplace and Z transforms from the continuous- and discrete-time Fourier transforms:

The Laplace Transform

(1) We begin with causal $f(t)$ and find its Fourier transform (Note that because $f(t)$ is causal, the integral has limits of 0 and ∞):

$$F(j\Omega) = \int_0^\infty f(t)e^{-j\Omega t}dt$$

(2) We note that for some functions $f(t)$ (for example the unit step function), the Fourier integral does not converge.

(3) We introduce a weighted function

$$w(t) = f(t)e^{-\sigma t}$$

and note

$$\lim_{\sigma\to 0} w(t) = f(t)$$

The effect of the exponential weighting by $e^{-\sigma t}$ is to allow convergence of the integral for a much broader range of functions $f(t)$.

(4) We take the Fourier transform of $w(t)$

$$\begin{aligned} W(j\Omega) = \tilde{F}(j\Omega|\sigma) &= \int_0^\infty \left(f(t)e^{-\sigma t}\right)e^{-j\Omega t}dt \\ &= \int_0^\infty f(t)e^{-(\sigma+j\Omega)}dt \end{aligned}$$

and define the complex variable $s = \sigma + j\Omega$ so that we can write

$$F(s) = \tilde{F}(j\omega|\sigma) = \int_0^\infty f(t)e^{-st}dt$$

$F(s)$ is the one-sided Laplace Transform. Note that the Laplace variable $s = \sigma + j\Omega$ is expressed in Cartesian form.

The Z transform

(1) We sample f(t) at intervals ΔT to produce $f^*(t)$. We take its Fourier transform (and use the sifting property of $\delta(t)$) to produce

$$F^*(j\Omega) = \sum_{n=0}^{\infty} f_n e^{-jn\Omega\Delta T}$$

(2) We note that for some sequences f_n (for example the unit step sequence), the summation does not converge.

(3) We introduce a weighted sequence

$$\{w_n\} = \{f_n r^{-n}\}$$

and note

$$\lim_{r\to 1}\{w_n\} = \{f_n\}$$

The effect of the exponential weighting by r^{-n} is to allow convergence of the summation for a much broader range of sequences f_n.

(4) We take the Fourier transform of w_n

$$\begin{aligned} W^*(j\Omega) = \tilde{F}^*(j\Omega|r) &= \sum_{n=0}^{\infty}\left(f_n r^{-n}\right)e^{-jn\Omega\Delta T} \\ &= \sum_{n=0}^{\infty} f_n\left(re^{j\Omega\Delta T}\right)^{-n} \end{aligned}$$

and define the complex variable $z = re^{j\Omega\Delta T}$ so that we can write

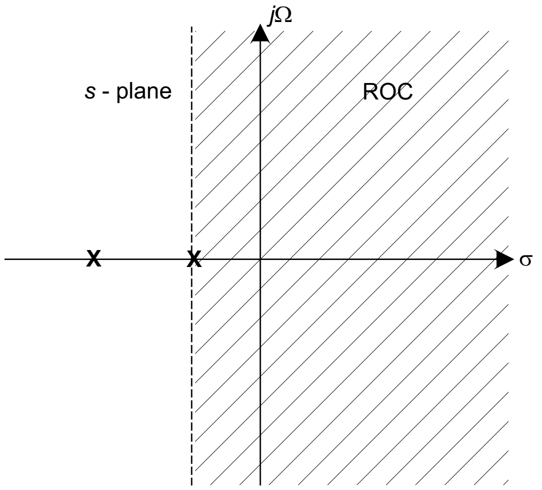
$$F(z) = \tilde{F}^*(j\Omega|r) = \sum_{n=0}^{\infty} f_n z^{-n}$$

$F(z)$ is the one-sided Z-transform. Note that $z = re^{j\Omega\Delta T}$ is expressed in polar form.

[1] D. Rowell October 22, 2008

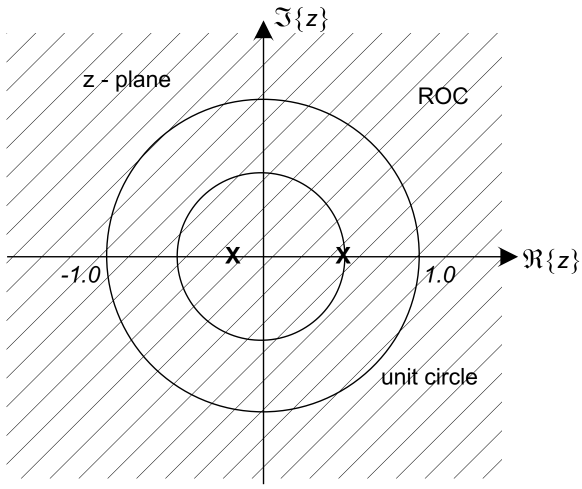
The Laplace Transform (contd.)

(5) For a causal function $f(t)$, the region of convergence (ROC) includes the s-plane to the right of all poles of $F(j\Omega)$.



The Z transform (contd.)

(5) For a right-sided (causal) sequence $\{f_n\}$ the region of convergence (ROC) includes the z-plane at a radius greater than all of the poles of $F(z)$.



We note that the mapping between the s plane and the z plane is given by

$$z = e^{s\Delta T}$$

and that the imaginary axis ($s = j\Omega$) in the s-plane maps to the unit circle ($z = e^{j\Omega\Delta T}$) in the z-plane.

Furthermore we note that the mapping of the unit circle in the z-plane to the imaginary axis in the s-plane is periodic with period 2π, and that the mapping of the $j\Omega$ axis to the unit circle produces aliasing for $|\Omega| > \pi/\Delta T$.

If we define a normalized discrete-time frequency that is independent of ΔT, that is

$$\omega = \Omega\Delta T \qquad \omega \leq \pi$$

we can make the following comparisons:

The Laplace Transform (contd.)

(6) If the ROC includes the imaginary axis, the FT of $f(t)$ is $F(j\Omega)$:

$$F(j\Omega) = F(s)\,|_{s=j\Omega}$$

(7) The convolution theorem states

$$f(t) \otimes g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau \overset{\mathcal{L}}{\Longleftrightarrow} F(s)G(s)$$

(8) For an LTI system with transfer function $H(s)$, the frequency response is

$$H(s)\,|_{s=j\Omega} = H(j\Omega)$$

if the ROC includes the imaginary axis.

The Z transform (contd.)

(6) If the ROC includes the unit circle, the DFT of $\{f_n\}$, $n = 0, 1, \ldots, N-1$. is $\{F_m\}$ where

$$F_m = F(z)\,|_{z=e^{j\omega_m}} = F(e^{j\omega_m}),$$

where $\omega_m = 2\pi m/N$ for $m = 0, 1, \ldots, N-1$.

(7) The convolution theorem states

$$\{f_n\} \otimes \{g_n\} = \sum_{m=-\infty}^{\infty} f_m g_{n-m} \overset{\mathcal{Z}}{\Longleftrightarrow} F(z)G(z)$$

(8) For a discrete LSI system with transfer function $H(z)$, the frequency response is

$$H(z)\,|_{z=e^{j\omega}} = H(e^{j\omega}) \quad |\omega| \leq \pi$$

if the ROC includes the unit circle.

The Laplace Transform (contd.)

(9) The transfer function

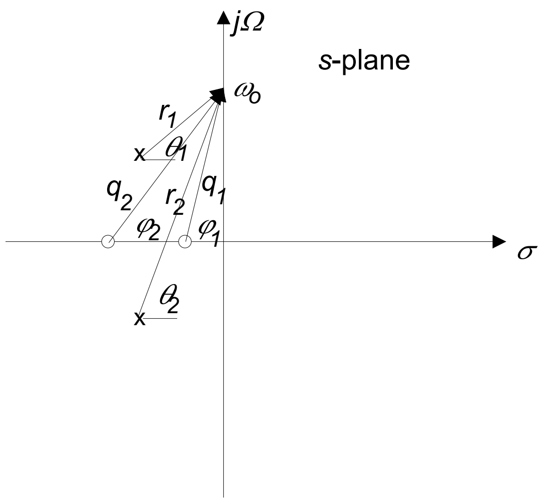
$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

is derived from the ordinary *differential* equation

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} \ldots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m f}{dt^m} + \ldots + b_1 \frac{df}{dt} + b_0 f$$

(10) Poles of $H(s)$ in the rh-plane indicate instability in the continuous-time system.

(11) The frequency response $H(j\omega)$ may be interpreted geometrically from the poles and zeros of $H(s)$ according to the following diagram:



then $H(j\Omega_o) = H(s)|_{s=j\Omega_o}$ and

$$|H(j\Omega)| = K \frac{\prod_{i=1}^{m} q_i}{\prod_{i=1}^{n} r_i}$$

$$\angle H(j\Omega) = \sum_{i=1}^{n} \phi_i - \sum_{i=1}^{n} \theta_i$$

(12) If $f(t) \overset{\mathcal{F}}{\Longleftrightarrow} F(s)$ then

$$f(t-\tau) \overset{\mathcal{F}}{\Longleftrightarrow} e^{-s\tau} F(s),$$

which is the delay property of the Laplace transform.

The Z transform (contd.)

(9) The transfer function

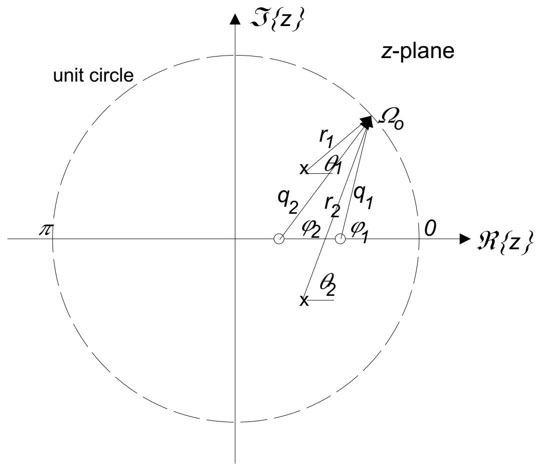
$$H(z) = \frac{b_m z^{-m} + b_{m-1} z^{-(m-1)} + \ldots + b_1 z^{-1} + b_0}{a_n z^{-n} + a_{n-1} z^{-(n-1)} + \ldots + a_1 z^{-1} + a_0}$$

is derived from the *difference* equation

$$a_0 y_k + a_1 y_{k-1} + \ldots + a_{n-1} y_{k-(n-1)} + a_0 y_{k-n} = b_0 f_k + b_1 f_{k-1} + \ldots + b_m f_{k-m}$$

(10) Poles of $H(z)$ outside the unit circle indicate instability in the discrete-time system.

(11) The frequency response $H(j\omega)$, $(\omega = \Omega/\Delta T)$ may be interpreted geometrically from the poles and zeros of $H(z)$ according to the following diagram:



then $H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$ and

$$\left|H(e^{j\omega})\right| = K \frac{\prod_{i=1}^{m} q_i}{\prod_{i=1}^{n} r_i}$$

$$\angle H(e^{j\omega}) = \sum_{i=1}^{n} \phi_i - \sum_{i=1}^{n} \theta_i$$

(12) If $\{f_n\} \overset{\mathcal{Z}}{\Longleftrightarrow} F(z)$ then

$$\{f_{n-m}\} \overset{\mathcal{Z}}{\Longleftrightarrow} z^{-m} F(z),$$

which is the delay (shift) property of the Z transform.