2.161 Signal Processing: Continuous and Discrete
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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING
2.161 Signal Processing – Continuous and Discrete

Introduction to Two-Dimensional/Image Processing [1]

1 Introduction

In this set of notes we extend the concepts of one-dimensional signal processing to two dimensions. Typical fields that involve the topics introduced here are imaging, including optics, and image processing.

The most important concept to grasp is that we are no longer dealing with functions of time, and the primary domain is the spatial domain. Therefore, instead of dealing with $f(t)$ we deal with $f(x)$, where x is a spatial distance, or in the two-dimensional situation $f(x, y)$. The methods developed here use Fourier techniques to transform the spatial representation $f(x, y)$ to a frequency domain space $F(\mathrm{j}\, u, \mathrm{j}\, v)$, where u, and v are *spatial frequencies* with units of radians/unit distance. As in temporal processing the difficult operation of convolution is replaced by simple multiplication in the frequency domain. In addition, an understanding of Fourier methods gives qualitative insights to image processing techniques such as "sharpening" and "blurring".

In these notes we begin by thoroughly examining one-dimensional Fourier methods and properties to gain insights on the Fourier transform and frequency domain processing, and then generalize to two-dimensional imaging.

2 Two-dimensional Imaging

We now address the more usual situation of processing a two-dimensional object function $f(x, y)$ through a linear image processing system, producing a two-dimensional image $g(x, y)$. The sifting property of the Dirac delta function may be extended to two dimensions:

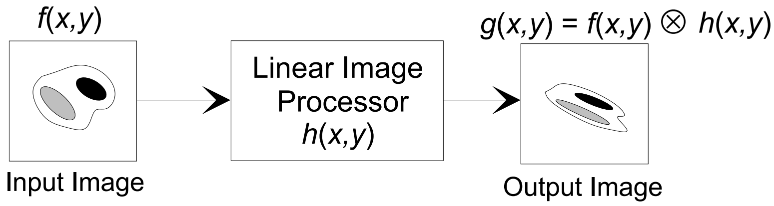


Figure 1: A two-dimensional linear image processing system.

$$\int\int_{-\infty}^{\infty} f(x, y)\delta(x - a)\delta(y - b)dxdy = f(a, b) \tag{1}$$

[1] D. Rowell, November 20, 2008

so that we can write the input $f(x,y)$ as a superposition of two-dimensional impulses $\delta^2(x,y) = \delta(x)\delta(y)$:

$$f(x,y) = \int\int_{-\infty}^{\infty} O(\sigma,\rho)\delta(x-\sigma)\delta(y-\rho)d\sigma d\rho \quad (2)$$

Assume that the action of the image processor can be expressed as a two-dimensional linear system operator $Ł_2\{\}$ that maps the input image $f(x,y)$ to the output image $g(x,y)$, that is

$$g(x,y) = Ł_2\{f(x,y)\} \quad (3)$$

so that we can write the image as

$$g(x,y) = Ł_2\left\{\int\int_{-\infty}^{\infty} f(\sigma,\rho)\delta(x-\sigma)\delta(y-\rho)d\sigma d\rho\right\} \quad (4)$$

As in one-dimensional signal processing, we can reverse the order of the operator and integration

$$g(x,y) = \int\int_{-\infty}^{\infty} f(\sigma,\rho)Ł_2\{\delta(x-\sigma)\delta(y-\rho)\}\, d\sigma d\rho \quad (5)$$

For a spatially invariant system, we define the *two-dimensional point-spread function* $h(x,y)$ as the *response to a two-dimensional delta function* $\delta^2(x,y) = \delta(x)\delta(y)$ at the origin of the object space:

$$h(x,y) = Ł_2\{\delta(x)\delta(y)\} \quad (6)$$

Equation (5) can be written

$$g(x,y) = \int\int_{-\infty}^{\infty} f(\sigma,\rho)h(x-\sigma,y-\rho)d\sigma d\rho \quad (7)$$

which is a two-dimensional convolution between the point-spread function $h(x,y)$ and the object space $f(x,y)$.

As in the one-dimensional case, we conclude that the point-spread function completely characterizes the imaging system, in the sense that the imaging fidelity is completely determined by $h(x,y)$.

■ Example 1

Determine the image formed by convolving a square uniform white image of size 20×20 by a linear image processing filter with point-spread function that is circularly symmetric with a radius of 5 units and height of unity.

Solution: The resulting images shown in Fig. 2 were computed using the following Matlab script:

```
% Matlab script to demonstrate 2-D convolution
% First create an array containing the point-spread function
psf = zeros(40,40);
for i=1:40
for j=1:40
   r = sqrt((i-20)^2 + (j-20)^2);
   if (r < 5)
     psf(i,j)= 1;
   end;
end;
end;
%
% Now create the object
object=zeros(40,40);
for i=10:29
for j=10:29
   object(i,j)=1;
end;
end;
%
% Set up a simple colormap so that all plots are in black and white
bw=[0 0 0];
colormap(bw);
%
% Make 3-D plots of the two input functions
mesh(psf);
title('Point-spread function');
pause; mesh(object);
title('Rectangular object');
pause;
%
% Use the conv2 function to do the 2-D convolution
image=conv2(psf,object,'same');
mesh(image);
title('Resulting image');
pause;
```

---

3 The Two-Dimensional Fourier Transform

In imaging processing work, representing two-dimensional functions, such as images, in the Fourier domain is of great importance. Let $f(x,y)$ be a function of two independent variables x and y; the Fourier transform of $f(x,y)$ is

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j(ux+vy)}dxdy \tag{8}$$

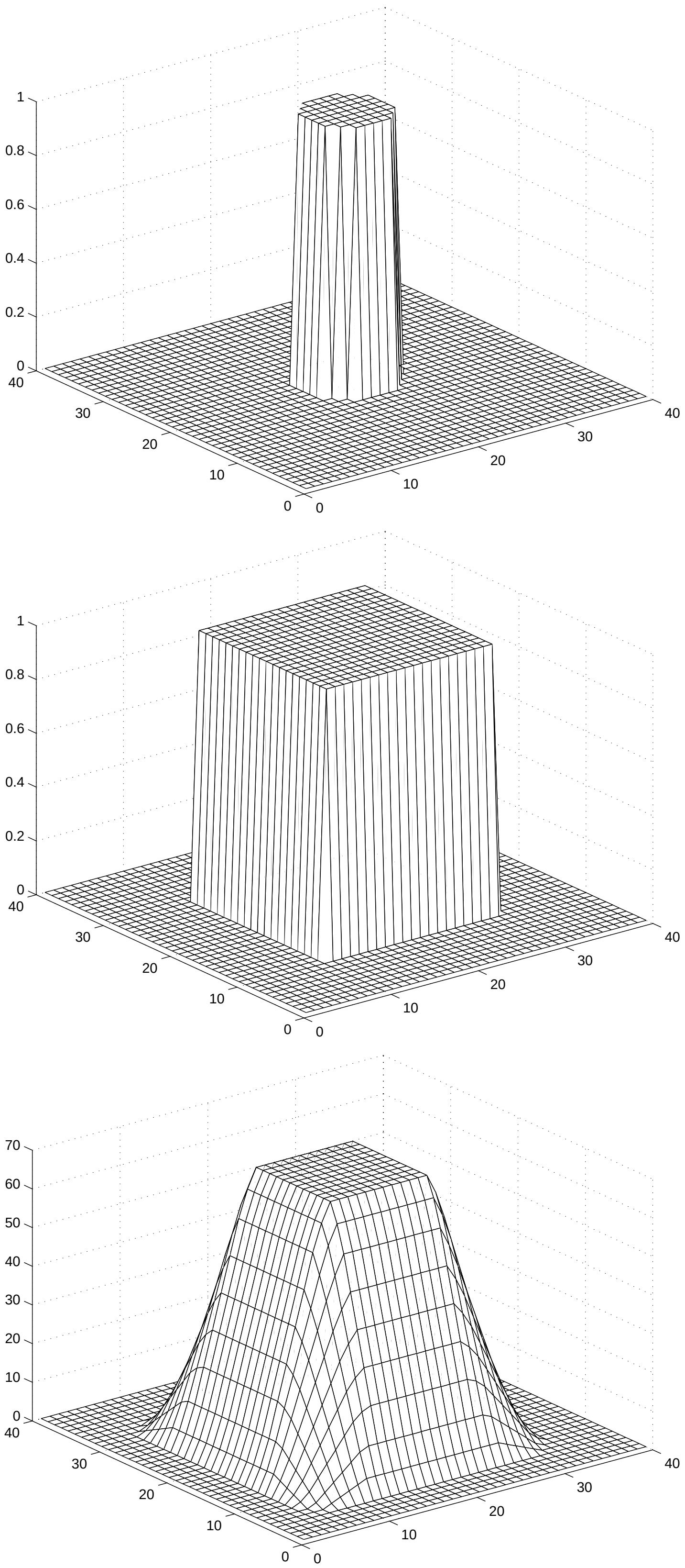


Figure 2: Two-dimensional convolution: (a) A circular point-spread function, ($r = 5$), (b) a uniform rectangular 20×20 object, and (c) the image formed by convolution.

where u and v are spatial frequencies with units of radians per unit length. By splitting the exponential into two halves it can be seen that the transform can be expressed as two one-dimensional transforms; first with respect to x and then with respect to y,

$$F(u,v) = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(x,y) e^{-jux} dx \right\} e^{-jvy} dy. \tag{9}$$

In general $F(u,v)$ is a complex-valued function of u and v.

■ Example 2

Find the two-dimensional Fourier transform of the two-dimensional impulse $f(x,y) = \delta^2(x,y) = \delta(x)\delta(y)$.

Solution: Separating the variables as in Eq. 9,

$$\Delta(u,v) = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \delta(x) e^{-jux} dx \right\} \delta(y) e^{-jvy} dy$$

and by the sifting property of the delta function,

$$\Delta(u,v) = 1$$

■ Example 3

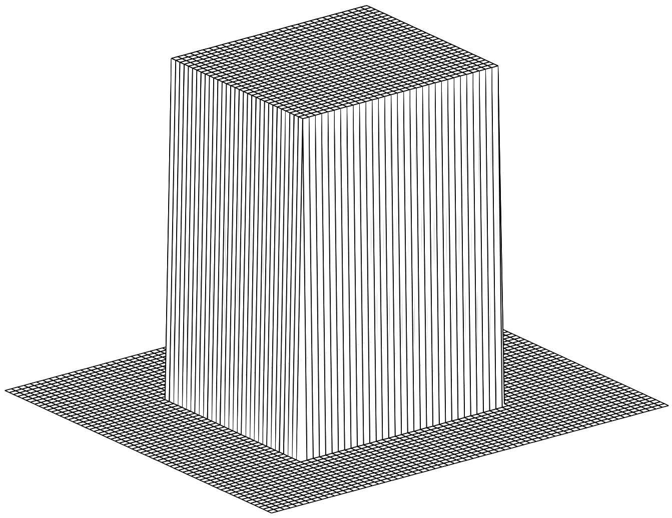
Find the Fourier transform of the function $f(x,y) = \text{rect}(x,y)$.
Solution: The function has a value of unity within the square defined by $-0.5 > x < 0.5, y > -0.5 < 0.5$. The transform is therefore

$$\begin{aligned} F(u,v) &= \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} e^{-j(ux+vy)} dx dy \\ &= \frac{\sin u}{u} \frac{\sin v}{v} = sinc(u,v) \end{aligned}$$

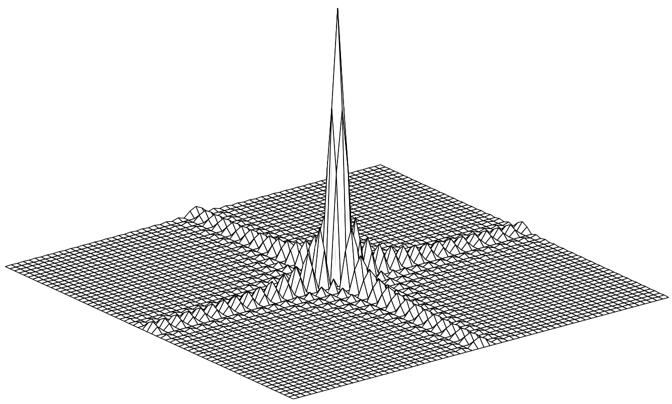
as shown in Fig. 3.

The two-dimensional inverse Fourier transform is

$$f(x,y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) e^{j(ux+vy)} du dv \tag{10}$$



(a)



(b)

Figure 3: The two-dimensional rect function and its Fourier Transform.

3.1 Properties of the Two-dimensional Fourier Transform

The following is list of some of the important properties of the transform. In all cases it is assumed

$$f(x,y) \overset{\mathcal{F}}{\Longleftrightarrow} F(u,v).$$

(1) Linearity:

$$af(x,y)+bg(x,y) \overset{\mathcal{F}}{\Longleftrightarrow} aF(u,v)+bG(u,v)$$

which follows directly from the linearity of the integration operation.

(2) Shift Property:

$$f(x-\alpha, y-\beta) \overset{\mathcal{F}}{\Longleftrightarrow} F(u,v)e^{-j(u\alpha+v\beta)}$$

which can be shown by a change in the variables of integration.

(3) Similarity (scaling):

$$f(\alpha x, \beta y) \overset{\mathcal{F}}{\Longleftrightarrow} \frac{1}{|\alpha\beta|}F\left(\frac{u}{\alpha}, \frac{v}{\beta}\right).$$

(4) Rotation by an Angle: we can write the transform in polar coordinates as

$$f(r,\theta) \overset{\mathcal{F}}{\Longleftrightarrow} F(\omega,\phi)$$

If the spatial domain function, f, is rotated by an angle α then the frequency domain function, F, is also rotated by the same angle:

$$f(r,\theta+\alpha) \overset{\mathcal{F}}{\Longleftrightarrow} F(\omega,\phi+\alpha)$$

(5) Convolution: The following result is analogous to the one dimensional case:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\sigma,\rho)g(x-\sigma,y-\rho)d\sigma d\rho \overset{\mathcal{F}}{\Longleftrightarrow} F(u,v)G(u,v)$$

so that convolution in the spatial domain is equivalent to multiplication in the frequency domain, or

$$f(x,y)\otimes g(x,y) \overset{\mathcal{F}}{\Longleftrightarrow} F(u,v)G(u,v)$$

The dual holds:

$$f(x,y)g(x,y) \overset{\mathcal{F}}{\Longleftrightarrow} \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\sigma,\rho)G(u-\sigma,v-\rho)d\sigma d\rho$$

(6) Parseval's Theorem:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\overline{g}(x,y)dxdy = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\overline{G}(u,v)dudv$$

where $\overline{f}$ denotes the complex conjugate. When $f(x,y)=g(x,y)$, we have Rayleigh's theorem:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)|^2\,dxdy = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(u,v)|^2\,dudv$$

which is a statement of the conservation of "energy" between the two domains.

3.2 Frequency domain description of system response

By analogy with the one-dimensional case, if a two-dimensional imaging system has a point-spread function $h(x, y)$ and its response to an object function $f(x, y)$ is $g(x, y)$, or

$$g(x, y) = f(x, y) \otimes h(x, y),$$

then in the frequency domain

$$G(u, v) = F(u, v)H(u, v) \tag{11}$$

where $H(u, v) = \mathcal{F}\{h(x, y)\}$, and is the two-dimensional *transfer function*. As in the one-dimensional case, the transfer function completely characterizes a linear imaging system.

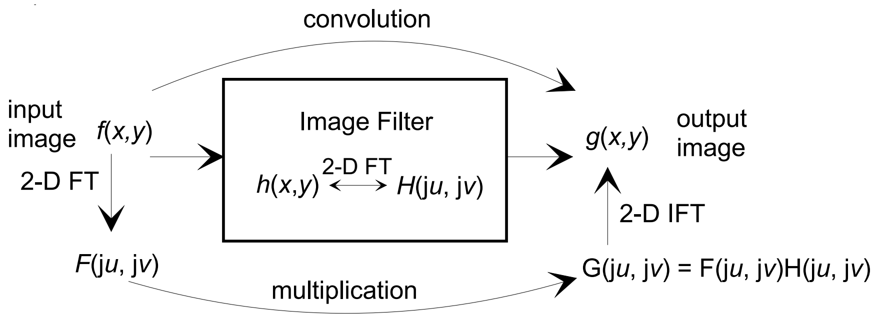


Figure 4: Two dimensional image filtering.

3.3 Frequency Domain Operations as a Filtering Process

The multiplication of a spectral representation $F(u, v)$ with a transfer function $H(u, v)$ may be considered as a frequency domain *filtering* operation in which regions of the input spectrum are emphasized or attenuated by the characteristics of the transfer function:

Low-pass filter: The low frequency region of the spectrum (adjacent to the origin) is emphasized, and the high frequency components are attenuated. This is generally a "blurring" operation that removes the fine detail in the image (and reduces noise).

High-pass filter The high frequency region of the spectrum (distant from the origin) is emphasized and the low frequency region attenuated. This results in a sharpening of the image in which discontinuities (edges) are enhanced, but noise in the image is also enhanced.

Band-Pass and Band-Reject Filters In these filters the transfer function is chosen to accentuate or attenuate an annular region of the input spectrum.

■ Example 4

Write a Matlab script to demonstrate the operations involved in two-dimensional image generation. Use the Fourier methods to convolve a circularly symmetric cylindrical ($r = 5$) point-spread function with a rectangular object.

Solution: The script is shown below.

```
% Matlab script to demonstrate 2-D convolution
% Set up a simple colormap so that all plots are in black and white
bw=[0 0 0];
colormap(bw);
%
% First create an array containing the circular point-spread
% function and compute its spectrum
%
psf = zeros(64,64);
for i=1:64
for j=1:64   r = sqrt((i-32)^2 + (j-32)^2);
   if (r < 5)
     psf(i,j)= 1;
   end;
end;
end;
fftpsf=fftshift(fft2(psf));
%
% Make 3-D plots of the PSF and its spectrum
%
figure(1);
mesh(psf);
title('Point-spread function');
figure(2);
mesh(abs(fftpsf));
title('Magnitude of FFT of PSF: Modulation Transfer Function');
pause;
%
% Now create the object
%
object=zeros(64,64);
for i=22:41
   for j=22:41
      object(i,j)=1;
   end;
end;
fftobj=fftshift(fft2(object));
```

```
%
% Plot these results
%
figure(1);
mesh(object);
title('Rectangular object');
figure(2);
mesh(abs(fftobj));
title('Magnitude of FFT of object');
pause;
%
% Use the ".*" operator to do the element by element
% frequency domain multiplication
%
fftimg = fftpsf.*fftobj;
image=fftshift(ifft2(fftimg));
%
% Plot the results
%
mesh(abs(fftimg));
title('Product of FFTs of PSF and object');
figure(1);
mesh(abs(image));
title('Resulting image');
```

The three-dimensional mesh plots produced by this script are shown in Figs. 5 and 6. In Fig. 5 the spatial domain functions of the psf, object and resulting image are shown, in Fig. 6 the equivalent spectral plots are shown.

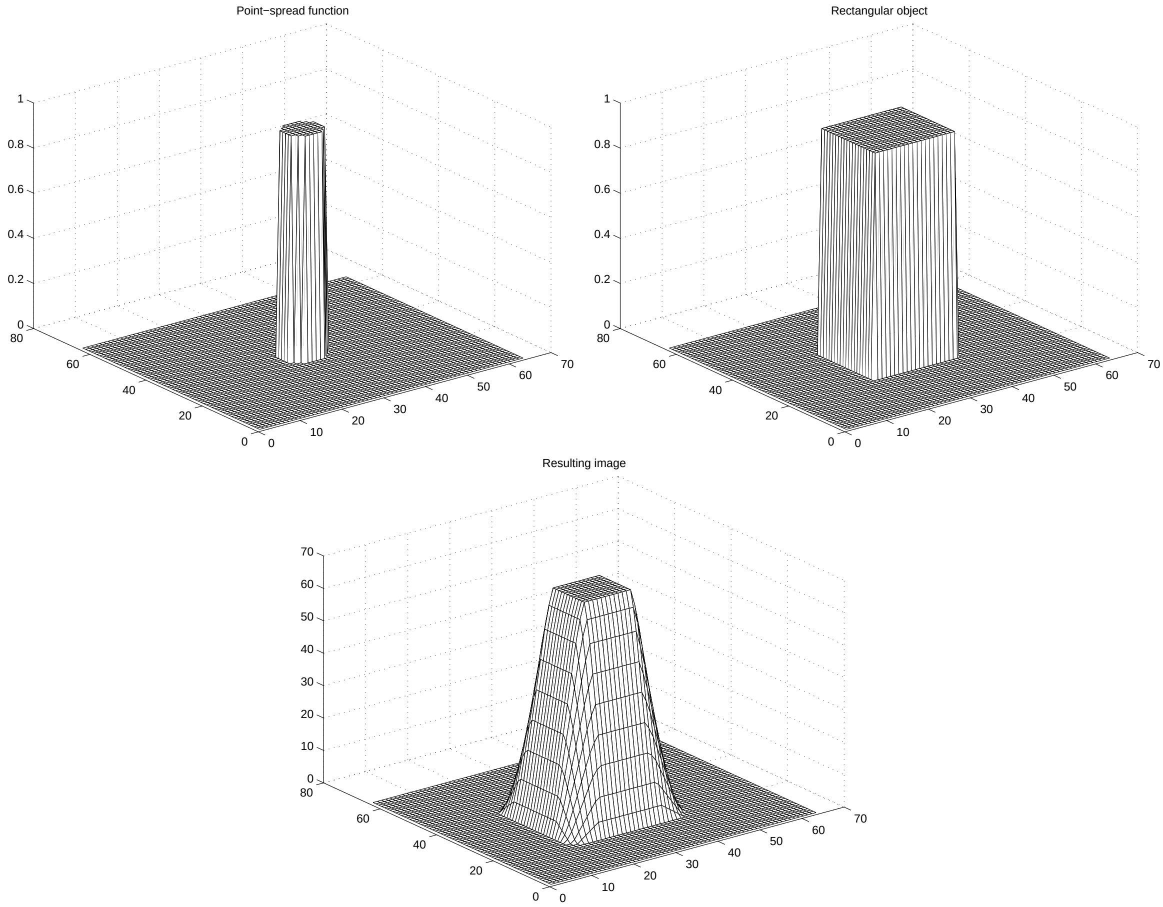


Figure 5: Frequency domain based convolution:(a) the point-spread function, (b) the object, and (c) the resulting image.

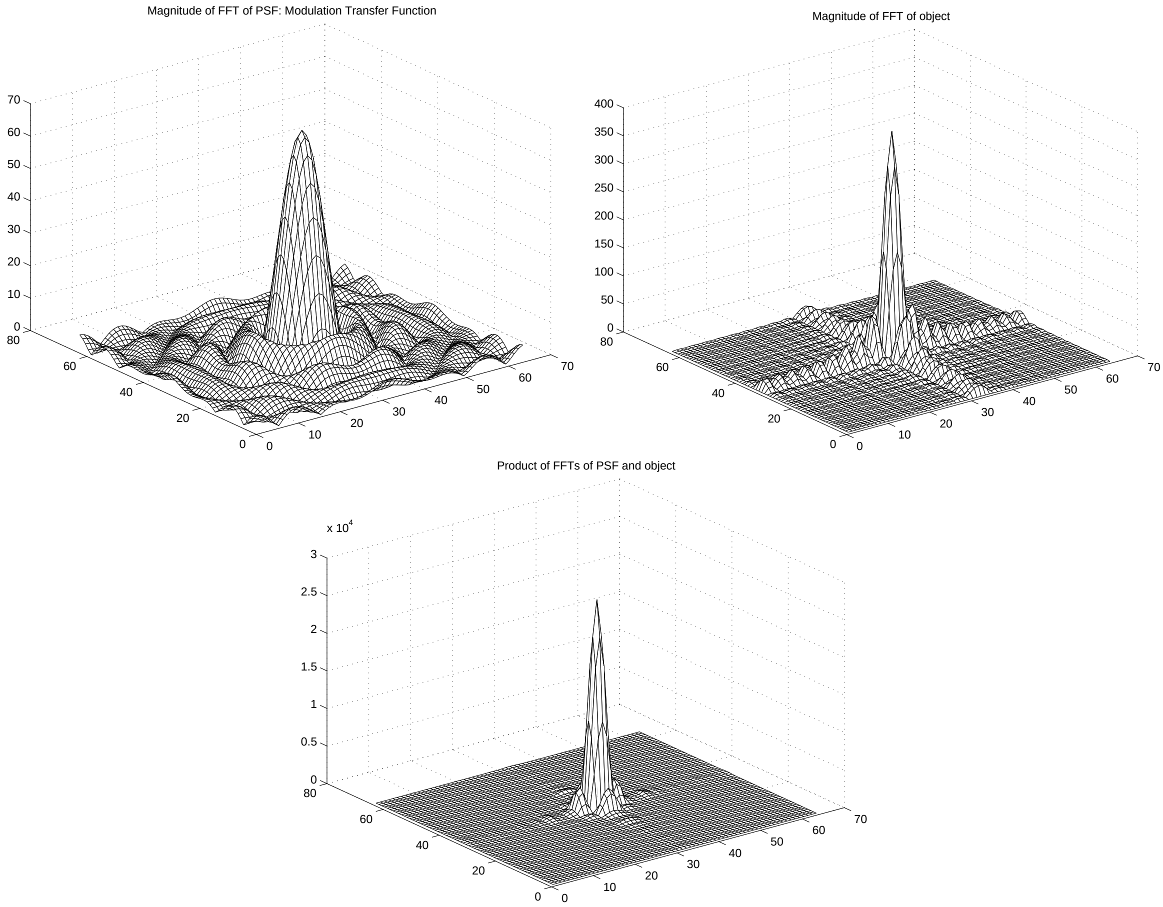


Figure 6: Frequency domain based convolution: the magnitude of the Fourier transforms of (a) the point-spread function, (b) the object, and (c) the resulting image.