2.161 Signal Processing: Continuous and Discrete
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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING
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Convolution[1]

1 Convolution

Consider a linear continuous-time LTI system with input $u(t)$, and response $y(t)$, as shown in Fig. 1. We assume that the system is initially at rest, that is all initial conditions are zero at time $t = 0$, and examine the time-domain forced response $y(t)$ to a continuous input waveform $u(t)$.

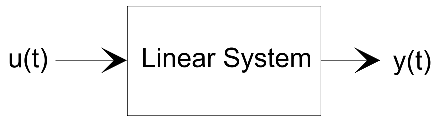

Figure 1: A linear system.

In Fig. 2 an arbitrary continuous input function $u(t)$ has been approximated by a *staircase* function $\tilde{u}_T(t) \approx u(t)$, consisting of a series of *piecewise constant* (zero order) sections each of an

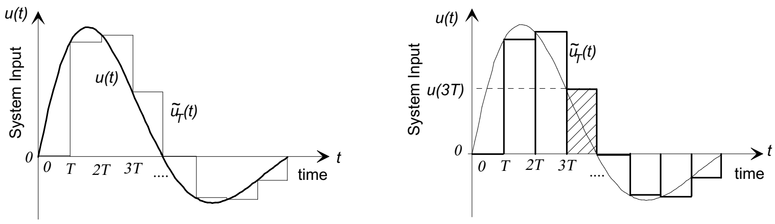

Figure 2: Staircase approximation to a continuous input function $u(t)$.

arbitrary fixed duration, T, where

$$\tilde{u}_T(t) = u(nT) \qquad \text{for } nT \leq t < (n+1)T \tag{1}$$

for all n. It can be seen from Fig. 2 that as the interval T is reduced, the approximation becomes more exact, and in the limit

$$u(t) = \lim_{T \to 0} \tilde{u}_T(t).$$

The staircase approximation $\tilde{u}_T(t)$ may be considered to be a sum of non-overlapping delayed pulses $p_n(t)$, each with duration T but with a different amplitude $u(nT)$:

$$\tilde{u}_T(t) = \sum_{n=-\infty}^{\infty} p_n(t) \tag{2}$$

where

$$p_n(t) = \begin{cases} u(nT) & nT \leq t < (n+1)T \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

[1] D. Rowell, September 8, 2008

Each component pulse $p_n(t)$ may be written in terms of a delayed unit pulse $\delta_T(t)$, of width T and amplitude $1/T$ that is:

$$p_n(t) = u(nT)\delta_T(t - nT)T \tag{4}$$

so that Eq. (2) may be written:

$$\tilde{u}_T(t) = \sum_{n=-\infty}^{\infty} u(nT)\delta_T(t - nT)T. \tag{5}$$

We now assume that the system response to an input $\delta_T(t)$ is a known function, and is designated $h_T(t)$ as shown in Fig. 3. Then if the system is linear and time-invariant, the response to a delayed unit pulse, occurring at time nT is simply a delayed version of the pulse response:

$$y_n(t) = h_T(t - nT) \tag{6}$$

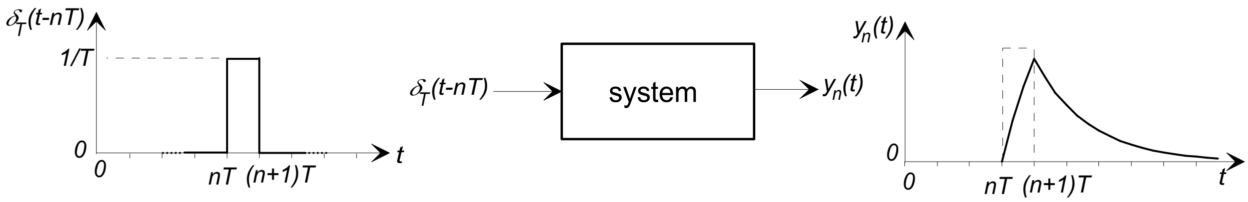

Figure 3: System response to a unit pulse of duration T.

The principle of superposition allows the total system response to $\tilde{u}_T(t)$ to be written as the sum of the responses to all of the component weighted pulses in Eq. (5):

$$\tilde{y}_T(t) = \sum_{n=-\infty}^{\infty} u(nT)h_T(t - nT)T \tag{7}$$

as shown in Fig. 4.

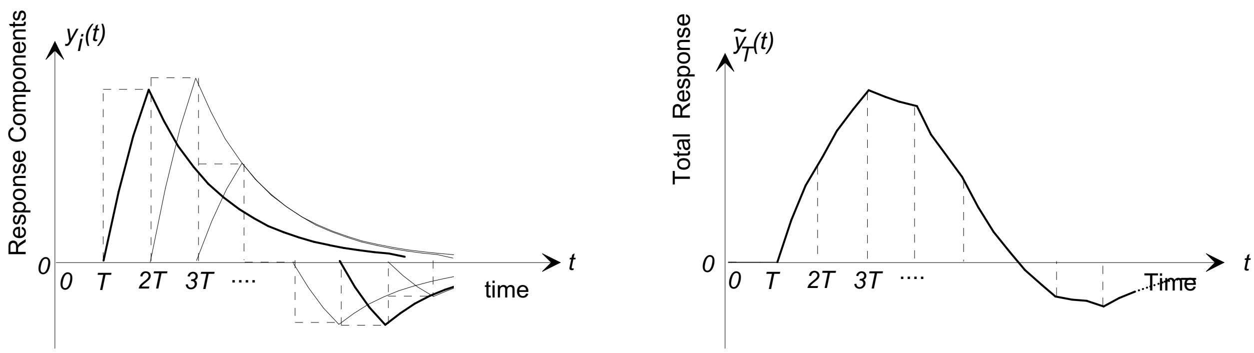

Figure 4: System response to individual pulses in the staircase approximation to $u(t)$.

For causal systems the pulse response $h_T(t)$ is zero for time $t < 0$, and future components of the input do not contribute to the sum, so that the upper limit of the summation may be rewritten:

$$\tilde{y}_T(t) = \sum_{n=-\infty}^{N} u(nT)h_T(t - nT)T \qquad \text{for } NT \leq t < (N+1)T. \tag{8}$$

Equation (8) expresses the system response to the staircase approximation of the input in terms of the system pulse response $h_T(t)$. If we now let the pulse width T become very small, and write $nT = \tau$, $T = d\tau$, and note that $\lim_{T\to 0} \delta_T(t) = \delta(t)$, the summation becomes an integral:

$$y(t) = \lim_{T\to 0} \sum_{n=-\infty}^{N} u(nT)h_T(t - nT)T \quad (9)$$

$$= \int_{-\infty}^{t} u(\tau)h(t - \tau)d\tau \quad (10)$$

where $h(t)$ is defined to be the system *impulse response*,

$$h(t) = \lim_{T\to 0} h_T(t). \quad (11)$$

Equation (10) is an important integral in the study of linear systems and is known as the *convolution* or *superposition* integral. It states that the system is entirely *characterized* by its response to an impulse function $\delta(t)$, in the sense that the forced response to any arbitrary input $u(t)$ may be computed from knowledge of the impulse response alone. The convolution operation is often written using the symbol $\otimes$:

$$y(t) = u(t) \otimes h(t) = \int_{-\infty}^{t} u(\tau)h(t - \tau)d\tau. \quad (12)$$

Equation (12) is in the form of a linear operator, in that it transforms, or maps, an input function to an output function through a linear operation.

The form of the integral in Eq. (10) is difficult to interpret because it contains the term $h(t-\tau)$ in which the variable of integration has been negated. The steps implicitly involved in computing the convolution integral may be demonstrated graphically as in Fig. 5, in which the impulse response $h(\tau)$ is reflected about the origin to create $h(-\tau)$, and then shifted to the right by t to form $h(t - \tau)$. The product $u(t)h(t - \tau)$ is then evaluated and integrated to find the response. This graphical representation is useful for defining the limits necessary in the integration. For example, since for a physical system the impulse response $h(t)$ is zero for all $t < 0$, the reflected and shifted impulse response $h(t - \tau)$ will be zero for all time $\tau > t$. The upper limit in the integral is then at most t. If in addition the input $u(t)$ is time limited, that is $u(t) \equiv 0$ for $t < t_1$ and $t > t_2$, the limits are:

$$y_f(t) = \begin{cases} \int_{t_1}^{t} u(\tau)h(t - \tau)d\tau & \text{for } t < t_2 \\ \int_{t_1}^{t_2} u(\tau)h(t - \tau)d\tau & \text{for } t \geq t_2 \end{cases} \quad (13)$$

■ Example

A simple RC first-order filter, shown in Fig. 6, is subjected to a very short unit impulsive voltage of duration $\Delta T = 0.001$ seconds and magnitude 10 volts, and is observed to respond with a output $v_o(t) = 0.03e^{-3t}$. Find the response of the filter to a ramp in applied voltage $V(t) = t$ for $t > 0$.

Solution: The product of the impulsive force and its duration $V\Delta T = 0.01$, and because of its brief duration, the pulse may be considered to be an impulse of strength

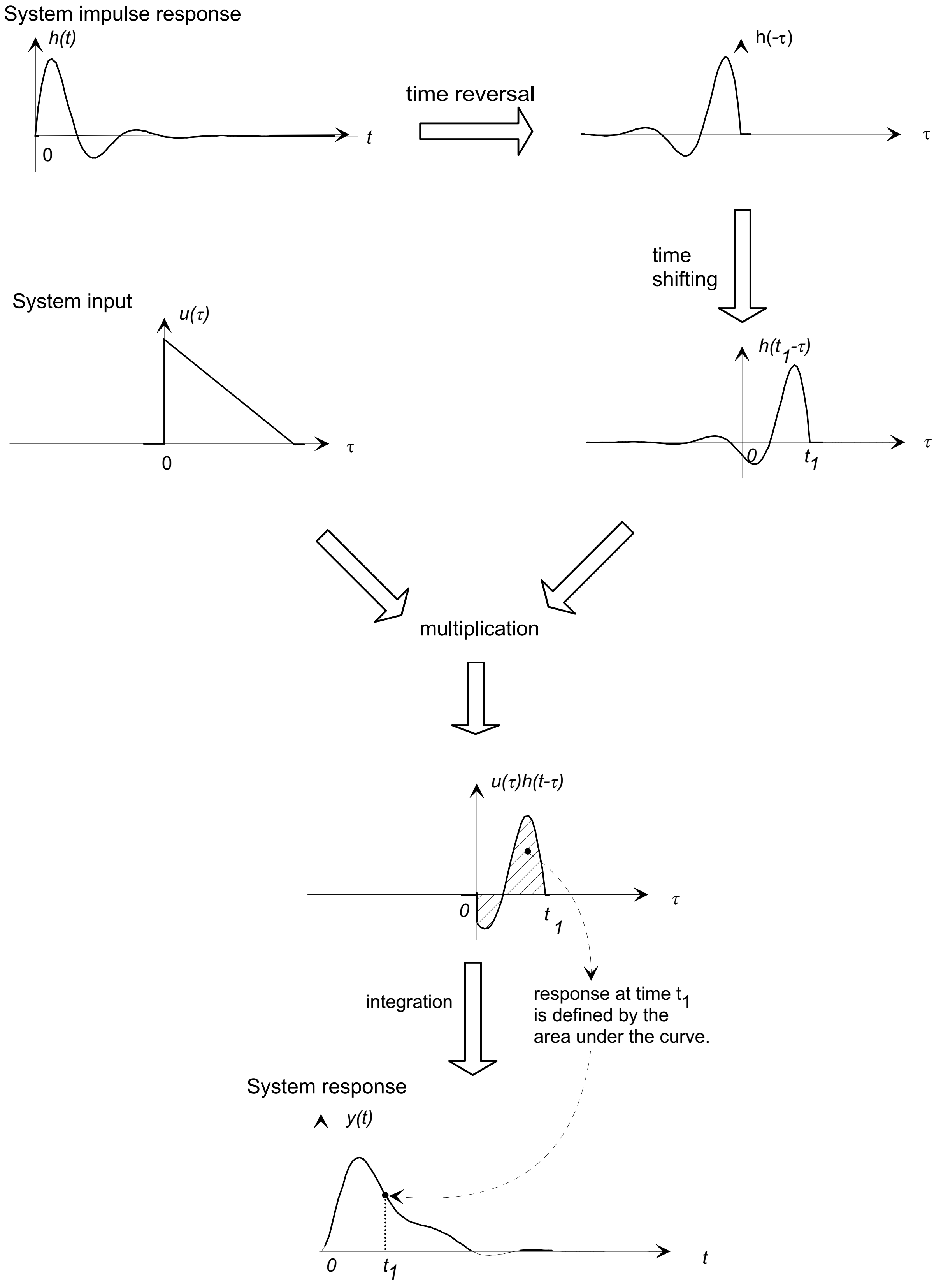

Figure 5: Graphical demonstration of the convolution integral.

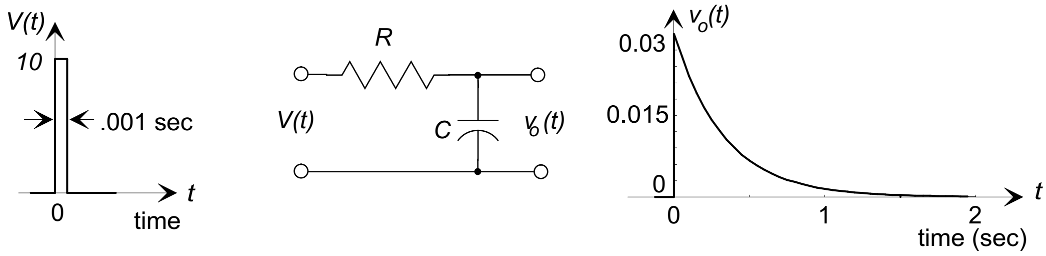

Figure 6: An RC filter and its impulse response.

0.01. The measured response $v_o(t)$ may then be taken as a scaled system impulse response $0.01h(t)$, and we assume that

$$h(t) = 3e^{-3t}. \tag{14}$$

The response to a ramp in input force, $F(t) = t$ for $t > 0$, may be found by direct substitution into the convolution integral using the assumed impulse response:

$$\begin{aligned} v_o(t) &= \int_0^t \tau 3e^{-3(t-\tau)} d\tau & (15)\\ &= 3e^{-3t} \int_0^t \tau e^{3\tau} d\tau & (16)\end{aligned}$$

where the limits have been chosen because the system is causal, and the input is identically zero for all $t < 0$. Integration by parts gives the solution

$$v(t) = t - \frac{1}{3} + \frac{1}{3}e^{-3t}. \tag{17}$$

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2 Properties

2.1 Linearity

Convolution is a linear operation and is commutative, associative and distributive, that is

$$\begin{aligned} u(t) \otimes h(t) &= h(t) \otimes u(t) && \text{(commutative)}\\ u(t) \otimes [h_1(t) \otimes h_2(t)] &= [u(t) \otimes h_1(t)] \otimes h_2(t) && \text{(associative)}\\ u(t) \otimes [h_1(t) + h_2(t)] &= [u(t) \otimes h_1(t)] + [u(t) \otimes h_2(t)] && \text{(distributive).}\end{aligned} \tag{18}$$

The associative property may be interpreted as an expression for the response on two systems in cascade or series, and indicates that the impulse response of two systems is $h_1(t) \otimes h_2(t)$, as shown in Fig. 7. Similarly the distributive property may be interpreted as the impulse response of two systems connected in parallel, and that the equivalent impulse response is $h_1(t) + h_2(t)$. The convolution operation is also associative with respect to scalar multiplication

$$a(h(t) \otimes u(t)) = (ah\,(t)) \otimes u(t) = h(t) \otimes (au\,(t))\,. \tag{19}$$

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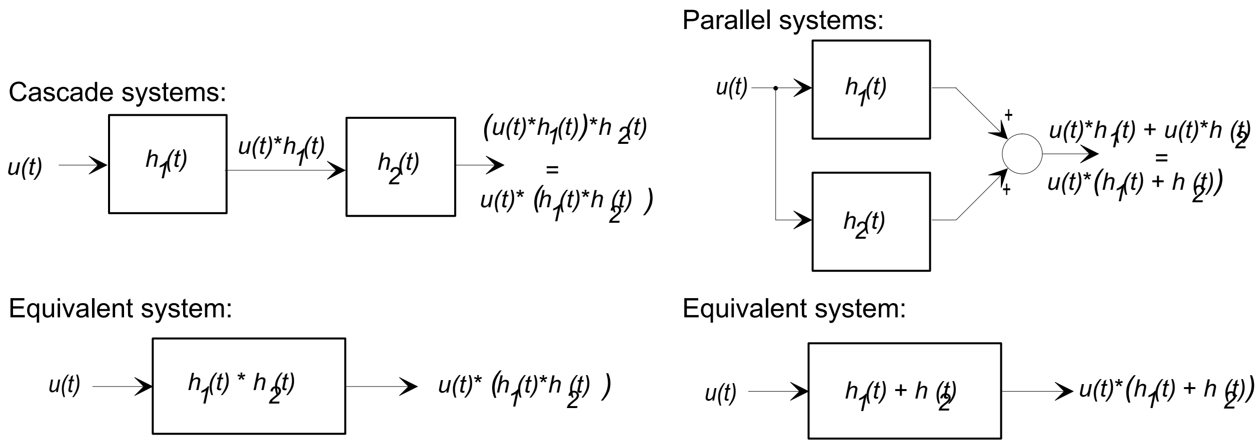

Figure 7: Impulse response of series and parallel connected systems.

2.2 Differentiation

$$\frac{d}{dt}\left(f(t)\otimes g(t)\right)=\frac{df}{dt}\otimes g(t)=f(t)\otimes\frac{dg}{dt} \tag{20}$$

2.3 Fourier Transform Relationships

Let $y(t)=f(t)\otimes g(t)$, then the Fourier transform $Y(j\Omega)=\mathcal{F}\left\{y(t)\right\}$ is

$$\begin{aligned} Y\left(j\Omega\right) &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}f(\tau)g(t-\tau)d\tau\right)e^{-j\Omega t}dt \\ &= \int_{-\infty}^{\infty}f(\tau)e^{-j\Omega\tau}d\tau.\int_{-\infty}^{\infty}g(\nu)e^{-j\Omega\nu}d\nu \\ &= F(j\Omega)G(j\Omega), \end{aligned} \tag{21}$$

which states that the Fourier transform of a convolution is the product of the component Fourier transforms.

Similarly[2]

$$\mathcal{F}^{-1}\left\{F(j\Omega)\otimes G(j\Omega)\right\}=\frac{1}{2\pi}\left(f(t)g(t)\right). \tag{22}$$

leading to the duality property that a convolution operation in the time domain is equivalent to a multiplicative operation in the frequency domain, and vice-versa.

[2]The appearance of the factor $1/2\pi$ depends on the definition of the Fourier transform. We assume here that $F(j\Omega)=\int_{-\infty}^{\infty}f(t)e^{-j\Omega t}dt$ and $f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(j\Omega)e^{j\Omega t}dt$.