2.161 Signal Processing: Continuous and Discrete
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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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Determining a System's Causality from its Frequency Response [1]

A causal system is one that is non-anticipatory, that is its impulse response $h(t) = 0$ for all time $t < 0$. Physical filters and systems, either electrical or mechanical, are always causal, but signal processing algorithms may be acausal, for example the simple FIR filter

$$y_n = f_{n+1} + f_n + f_{n-1}$$

obviously anticipates a future sample. The question addressed here is: what are the conditions on the frequency response function $H(j\Omega)$ of a filter that specify whether a system is causal or not?

We start with some simple observations based on the properties of the Fourier transform. Since $h(t) \stackrel{\mathcal{F}}{\Longleftrightarrow} H(j\Omega)$, and

(a) If $h(t)$ is real, $H(-j\Omega) = \overline{H(j\Omega)}$, where $\overline{H}$ indicates the complex conjugate.

(b) If $h(t)$ is real and even, then $H(j\Omega)$ is real and even.

(c) If $h(t)$ is real and odd, then $H(j\Omega)$ is imaginary and odd.

We can immediately conclude that if $H(j\Omega)$ is real and even, or $H(j\Omega)$ is imaginary and odd, that $h(t)$ will be two-sided (even or odd) and that the system will be acausal. Our first conclusion is therefore that in order for $h(t)$ to be causal, $H(j\Omega)$ must be complex and satisfy condition (a) above. "Ideal" filters, specified in terms of $|H(j\Omega)|$ alone, with zero phase response, are acausal on this basis alone.

We now investigate a relationship between $\Re\{H(j\Omega)\}$ and $\Im\{H(j\Omega)\}$ that will guarantee a causal system. The impulse response $h(t)$ can be decomposed into the sum of an *even* function $h_e(t)$, and an *odd* function $h_o(t)$

$$h(t) = h_e(t) + h_o(t) \tag{1}$$

where

$$h_e(t) = \frac{1}{2}\left(h(t) + h(-t)\right) \tag{2}$$

and

$$h_o(t) = \frac{1}{2}\left(h(t) - h(-t)\right). \tag{3}$$

For example, consider the simple causal first-order system with impulse response $h(t) = u_s(t)e^{-at}$, where $u_s(t)$ is the unit-step (Heaviside) function and $a > 0$. Then

$$\begin{aligned} h_e(t) &= \frac{1}{2}u_s(t)e^{-at} + \frac{1}{2}u_s(-t)e^{at} \\ h_o(t) &= \frac{1}{2}u_s(t)e^{-at} - \frac{1}{2}u_s(-t)e^{at} \end{aligned}$$

[1] D. Rowell January 27, 2009

In order that $h(t)$ be zero for $t < 0$, we require that

$$h_o(t) = \begin{cases} h_e(t), & t > 0 \\ -h_e(t), & t < 0 \end{cases} \tag{4}$$

$$= \operatorname{sgn}(t) h_e(t) \tag{5}$$

which is clearly the case in the above example. Then the causal impulse response may be written in terms of the even function alone

$$h(t) = h_e(t) + \operatorname{sgn}(t) h_e(t). \tag{6}$$

In the frequency domain, the frequency response function $H(j\Omega)$ can also be expressed in terms of the even function alone:

$$H(j\Omega) = \mathcal{F}\{h_e(t)\} + \frac{1}{2\pi}\left(\mathcal{F}\{\operatorname{sgn}(t)\} \otimes \mathcal{F}\{h_e(t)\}\right) \tag{7}$$

$$= H_e(j\Omega) - j\left[\frac{1}{\pi\Omega} \otimes H_e(j\Omega)\right] \tag{8}$$

where multiplication in the time domain has been replaced with convolution in the frequency domain, and $\mathcal{F}\{\operatorname{sgn}(t)\} = \mathcal{F}\{u_s(t) - u_s(-t)\} = 2/j\Omega$.

We digress briefly to introduce the Hilbert transform, $\hat{f}(t)$, defined as the convolution of $f(t)$ with $1/\pi t$, or

$$\widehat{f}(t) = f(t) \otimes \frac{1}{\pi t} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(\nu)}{(t-\nu)} d\nu \tag{9}$$

The Hilbert transform is used often in communication systems. In the frequency domain it has the interesting property of being a "phase shifter". From the convolution theorem

$$\mathcal{F}\left\{\widehat{f}(t)\right\} = \mathcal{F}\left\{\frac{1}{\pi t}\right\} \mathcal{F}\{f(t)\} = -j\operatorname{sgn}(\Omega) F(j\Omega). \tag{10}$$

Notice that positive-frequency components are multiplied by $-j$, and all negative-frequency components by j. The magnitude spectrum is unaltered by the Hilbert transform, but the phase is shifted by $\pi/2$ radians.

We now return to Eq. 8, where we can write the frequency response function in terms of its even part alone:

$$H(j\Omega) = H_e(j\Omega) - j\widehat{H_e}(j\Omega) \tag{11}$$

where $\widehat{H_e}(j\Omega)$ is the Hilbert transform of $H_e(j\Omega)$.

Equation 11 demonstrates some important properties of the frequency response of a causal system:

(a) The imaginary part of the frequency response of a causal linear system is determined by the Hilbert transform of the real part. In other words, knowledge of the real part is sufficient to completely specify the system, and the imaginary part is "redundant" information.

(b) A check for system causality is to compare the imaginary part of the frequency response with the Hilbert transform of the real part.

(c) The real part of the frequency response $\Re\{H(j\Omega)\}$ is the Fourier transform of the even part of the system impulse response.

(d) The imaginary part of the frequency response $\Im\{H(j\Omega)\}$ is the Fourier transform of the odd part of the system impulse response.