2.161 Signal Processing: Continuous and Discrete
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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING
2.161 *Signal Processing - Continuous and Discrete*

Sinusoidal Frequency Response of Linear Systems[1]

1 Introduction

In this class note we examine the response of linear systems to sinusoidal inputs. As we have discussed, Fourier analysis allows any physical repetitive waveform may be represented by an infinite sum of harmonically related sinusoids, therefore knowledge of the system response to a sinusoidal input provides a basis for determining the response to a broad class of periodic inputs. The sinusoidal waveform is described by

$$u(t) = A\sin(\Omega t + \psi) \tag{1}$$

where A is the amplitude of the input, Ω is the angular frequency, and ψ is the phase. This waveform is periodic, with period $T = 2\pi/\Omega$.

2 The Steady-State Frequency Response

In analyzing system response to sinusoidal inputs of the form of Eq. (1), it is generally assumed that the input $u(t)$ has existed for all time t, and the solution is sought for the "steady-state" periodic response after all transient terms in the response have decayed to insignificance. The response of a stable linear system to an input $u(t)$ is the sum of a homogeneous component, and a particular solution $y_p(t)$:

$$y(t) = \sum_{i=1}^{n} C_i e^{\lambda_i t} + y_p(t) \tag{2}$$

where n is the system order, the λ_i are the roots (assumed to be distinct) of the characteristic equation, and C_i are n constants to be determined from the initial conditions. For a stable system all terms $e^{\lambda_i t}$ will ultimately decay to zero.

If the input $u(t)$ is periodic with period T, that is $u(t) = u(t+T)$ for all t, the particular response $y_p(t)$ is also periodic and therefore persists for all time. For a stable system it is convenient to consider the total response $y(t)$ in two regions, an initial *transient* region in which the exponential homogeneous components $e^{\lambda_i t}$ must be considered, followed by the *steady-state* region when the homogeneous solution components have all decayed to the point of becoming insignificant, as shown in Fig. 1. In the steady-state region, only the particular response component $y_p(t)$ of the response is considered. The sinusoidal system response is defined in the steady-state region

$$y_{ss}(t) = \lim_{t\to\infty} y(t) = y_p(t). \tag{3}$$

For a linear system characterized by a transfer function $H(s)$, the particular response component $y_p(t)$ to an exponential input $u(t) = U(s)e^{st}$ is:

$$y_p(t) = Y(s)e^{st} = H(s)U(s)e^{st}. \tag{4}$$

[1] D. Rowell, August 15, 2008

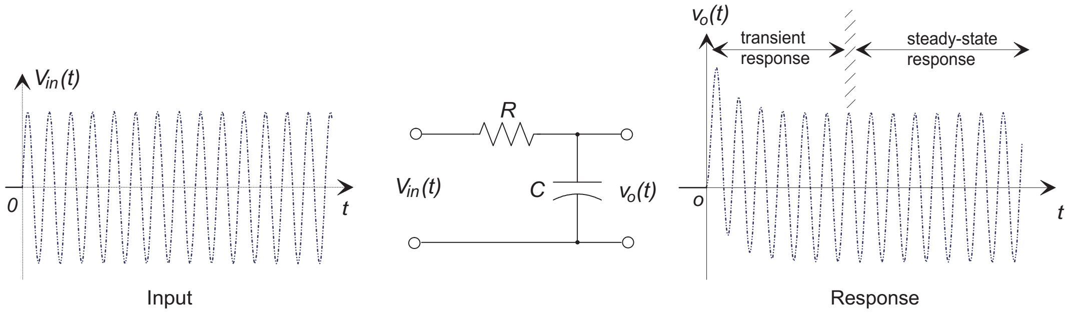

Figure 1: Transient and steady-state response of a linear system to a sinusoidal input beginning at time $t = 0$.

The transfer function $H(s)$ provides the basis for the development of the steady-state response to sinusoidal inputs.

3 The Complex Frequency Response

The response of a linear system to the complex exponential waveform $u(t) = U(s)e^{st}$, may be extended to describe the response to sinusoidal inputs of the form of Eq. (1). In order to derive the sinusoidal response from the exponential response we initially set s to be purely imaginary, that is $s = j\Omega$, so that the input takes the form of a general complex sinusoid:

$$u(t) = U(s)e^{st}\Big|_{s=j\Omega} = U(j\Omega)e^{j\Omega t} \tag{5}$$

The Euler relationships allow complex exponentials to be written as complex sine and cosine functions:

$$e^{j\Omega t} = \cos\Omega t + j\sin\Omega t \tag{6}$$

$$e^{-j\Omega t} = \cos\Omega t - j\sin\Omega t. \tag{7}$$

With the substitution $s = j\Omega$, the steady-state exponential response $y_{ss}(t) = y_p(t)$, as seen from Eq. (4), is also a complex sinusoid with the angular frequency $j\Omega$:

$$y_{ss}(t) = Y(j\Omega)e^{j\Omega t} = H(j\Omega)U(j\Omega)e^{j\Omega t} \tag{8}$$

so that the output amplitude $Y(j\Omega)$ is

$$Y(j\Omega) = H(j\Omega)\,U(j\Omega)\,. \tag{9}$$

The function $H(j\Omega)$ is defined to be the *frequency response* of the system, and is related directly to the system transfer function:

$$H(j\Omega) = H(s)|_{s=j\Omega} = \frac{Y(j\Omega)}{U(j\Omega)}. \tag{10}$$

The transfer function of a linear system is a rational function in the complex variable s, expressed as the ratio of a numerator polynomial $N(s)$ and a denominator polynomial $D(s)$:

$$H(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}. \tag{11}$$

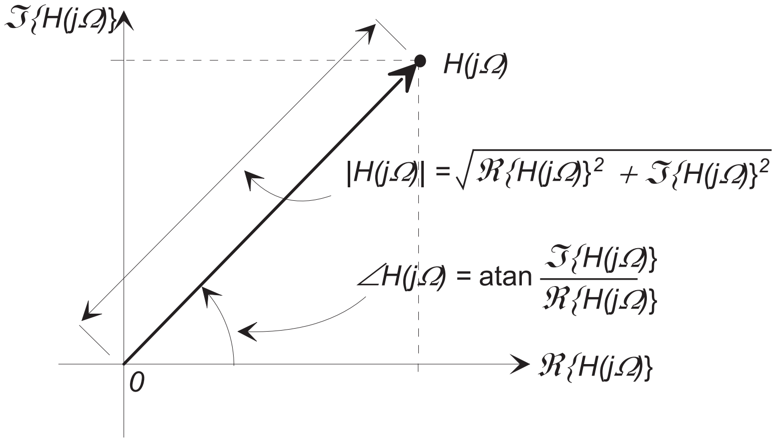

Figure 2: Definition of the magnitude and the phase angle of the complex frequency response.

The frequency response function $H(j\Omega)$ is therefore a similar rational function in the variable $j\Omega$:

$$H(j\Omega) = \frac{N(j\Omega)}{D(j\Omega)} = \frac{b_m(j\Omega)^m + b_{m-1}(j\Omega)^{m-1} + \ldots + b_1(j\Omega) + b_0}{a_n(j\Omega)^n + a_{n-1}(j\Omega)^{n-1} + \ldots + a_1(j\Omega) + a_0} \tag{12}$$

The frequency response $H(j\Omega)$ is a complex function, which can be expressed in terms of its real and imaginary parts

$$H(j\Omega) = \Re\{H(j\Omega)\} + \Im\{H(j\Omega)\} \tag{13}$$

where $\Re\{\}$ and $\Im\{\}$ are the real and imaginary operators and extract the real and imaginary parts of a complex expression.

It is useful to represent $H(j\Omega)$ in polar form, in terms of a magnitude function and phase function, as shown in Fig. 2:

$$H(j\Omega) = |H(j\Omega)|\, e^{\phi(j\Omega)} \tag{14}$$

where $|H(j\Omega)|$ is the *magnitude* of the frequency response given by

$$|H(j\Omega)| = \sqrt{\left(\Re\{H\,(j\Omega)\}\right)^2 + \left(\Im\{H\,(j\Omega)\}\right)^2}, \tag{15}$$

and $\phi(j\Omega)$ is the *phase angle* of the frequency response,

$$\phi(j\Omega) = \angle H(j\Omega) = \tan^{-1}\left(\frac{\Im\{H\,(j\Omega)\}}{\Re\{H\,(j\Omega)\}}\right). \tag{16}$$

Since $H(j\Omega)$ is complex, we can also define its complex conjugate $\overline{H(j\Omega)}$ as

$$\begin{aligned} \overline{H(j\Omega)} &= \Re\{H(j\Omega)\} - \Im\{H(j\Omega)\} & (17)\\ &= |H(j\Omega)|\, e^{-\phi(j\Omega)}. & (18) \end{aligned}$$

Examination of $N(j\Omega)$ and $D(j\Omega)$ in Eq. (12) shows that the real part of each polynomial comes from the even powers of $j\Omega$, while the imaginary part comes from the odd powers. Furthermore when $N(-j\Omega)$ and $D(-j\Omega)$ are evaluated, the real parts are the same as those of $N(j\Omega)$ and $D(j\Omega)$ while the imaginary parts are negated, with the result that

$$\begin{aligned} H(-j\Omega) &= \frac{N(-j\Omega)}{D(-j\Omega} = \frac{\overline{N(j\Omega)}}{\overline{D(j\Omega)}} \\ &= \overline{H(j\Omega)} \\ &= |H(j\Omega)|\, e^{-\phi(j\Omega)}. & (19) \end{aligned}$$

4 The Sinusoidal Frequency Response

The steady-state response of a linear single-input, single-output system to a real sinusoidal input of the the form of Eq. (1), that is $u(t) = A\sin(\Omega t + \psi)$ where A is the amplitude of the input and ψ is an arbitrary phase angle, is found directly from the system complex frequency response function $H(j\Omega)$. The Euler formulas (Eqs. (6) and (7) may be rearranged to allow real sinusoidal and cosinusoidal waveforms to be expressed as the sum of two complex exponentials:

$$\begin{aligned} \sin(\Omega t) &= \frac{1}{2j}(e^{j\Omega t} - e^{-j\Omega t}) && (20) \\ \cos(\Omega t) &= \frac{1}{2}(e^{j\Omega t} + e^{-j\Omega t}). && (21) \end{aligned}$$

so that the system input may be written

$$\begin{aligned} u(t) &= A\sin(\Omega t + \psi) \\ &= \frac{A}{2j}\left(e^{j(\Omega t+\psi)} - e^{-j(\Omega t+\psi)}\right). && (22) \end{aligned}$$

Equation (22) shows that the real input $u(t)$ can be expressed as the sum of two complex exponential components, $u_1(t) = (A/2j)e^{j(\Omega t+\psi)}$ and $u_2(t) = (-A/2j)e^{-j(\Omega t+\psi)}$. The principle of superposition allows the sinusoidal response to be written as the sum of the responses to the two complex exponential components:

$$\begin{aligned} y_{ss}(t) &= y_{ss1}(t) + y_{ss2}(t) \\ &= \frac{A}{2j}H(j\Omega)e^{j(\Omega t+\psi)} - \frac{A}{2j}H(-j\Omega)e^{-j(\Omega t+\psi)}. && (23) \end{aligned}$$

If $H(j\Omega)$ is written in its polar form (Eq. (18)) and $H(-j\Omega)$ is described by Eq. (19), Eq. (23) becomes

$$\begin{aligned} y_{ss}(t) &= \frac{A}{2j}\left|H(j\Omega)\right|\left(e^{j(\Omega t+\psi)}e^{j\phi(j\Omega)} - e^{-j\Omega t}e^{-j(\phi(j\Omega)+\psi)}\right) \\ &= A\left|H(j\Omega)\right|\frac{1}{2j}\left(e^{j(\Omega t+\psi+\phi(j\Omega))} - e^{-j(\Omega t+\psi+\phi(j\Omega))}\right) \\ &= A\left|H(j\Omega)\right|\sin(\Omega t + \psi + \phi(j\Omega)). && (24) \end{aligned}$$

The steady-state sinusoidal response is a sinusoidal function of the same angular frequency Ω as the input, but modified in its amplitude by the factor $|H(j\Omega)|$, and shifted in phase by the quantity $\phi(j\Omega)$. Thus, in general, the steady-state response of a linear SISO system to a sinusoidal input $u(t) = A\sin\Omega t$ can be characterized in terms of the magnitude of the frequency response function $|H(j\Omega)|$, and the phase shift $\phi(j\Omega) = \angle H(j\Omega)$. With knowledge of $|H(j\Omega)|$ and $\phi(j\Omega)$, the response may be determined directly from Eq. (24). Figure 3 shows a typical steady-state sinusoidal input and output of a linear system, demonstrating the modification to the amplitude and phase. The magnitude of the frequency response represents the *ratio* of the output amplitude to the input amplitude as a function of frequency. and is known as the *gain* of the system. A system that responds to low frequency inputs but attenuates high frequency inputs is known as a *low-pass* system, while a system that does not respond to low frequencies but responds to high frequencies is known as a *high-pass* system.

The phase angle $\phi(j\Omega)$ represents the temporal shift of the response sinusoid relative to the input, measured in either degrees or radians. If $\phi(j\Omega) < 0$ the system is said to exhibit a phase "lag" at that frequency, because the output waveform effectively lags behind the input. On the other hand if $\phi(j\Omega) > 0$ the system is said to exhibit a phase "lead".

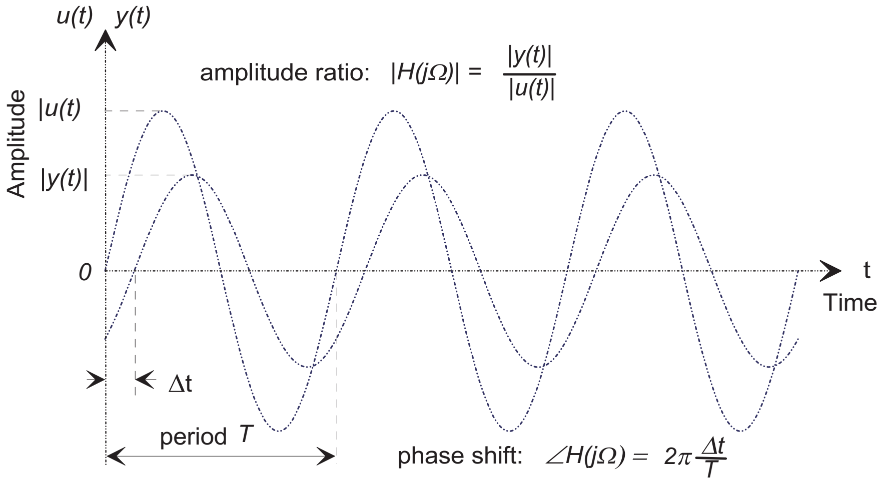

Figure 3: Steady-state response of a linear system with a sinusoidal forcing function.

■ Example 1

Capacitors are frequently used in electronic circuits to pass ac signals while "blocking" the dc voltages necessary for the operation of the transistors. Figure 4(a) shows a capacitor C being used to couple a microphone to an emitter follower transistor stage. The two resistors R_1 and R_2 are used to set the dc voltage of the base of the transistor at 6v (requiring $R_1 = R_2$). Assume $C = 0.1$ μfd, $R_1 = R_2 = 20$ kΩ. The task is to determine the effect of the components C, R_1, R_2 on the frequency response of the audio system

Solution: The equivalent circuit of the input components (C, R_1, R_2) is shown in Fig.

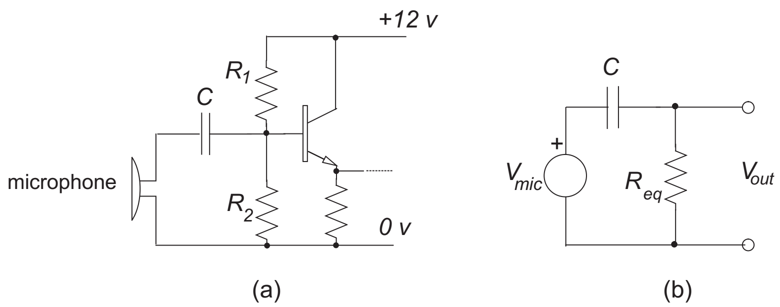

Figure 4: (a) A microphone amplifier and (b) the equivalent circuit of its input coupling components.

4(b). For ac signals it is assumed that the +12 volt supply is at ground potential, and the two resistors R_1 and R_2 are combined into an equivalent parallel resistance R_{eq}. The microphone is modeled as a voltage source. The transfer function of Fig. 4(b) is

$$H(s) = \frac{R_{eq}Cs}{R_{eq}Cs + 1}. \qquad \text{(i)}$$

The system frequency response is therefore

$$H(j\Omega) = H(s)|_{s=j\Omega} = \frac{jR_{eq}C\Omega}{1 + jR_{eq}C\Omega} \tag{ii}$$

and the magnitude and phase functions are:

$$|H(j\Omega)| = \frac{R_{eq}C\Omega}{\sqrt{1 + (R_{eq}C\Omega)^2}} \tag{iii}$$

$$\angle H(j\Omega) = \tan^{-1}\left(\frac{1}{R_{eq}C\Omega}\right). \tag{iv}$$

We note that $R_{eq} = R_1R_2/(R_1 + R_2) = 10\ \text{k}\Omega$. The following table shows the values of $|H(j\Omega)|$ and $\angle H(j\Omega)$ computed from Eqs. (iii) and (iv) for several frequencies:

Frequency (f) (Hz)	$\Omega = 2\pi f$ (rad/s)	$\lvert H(j\Omega)\rvert$	$\angle H(j\Omega)$ (rad)
0	0	0	1.57
1	6.28	0.006	1.56
10	62.8	0.062	1.50
50	314	0.299	1.27
250	1570	0.843	0.57
1000	6283	0.987	0.157
3000	1884	0.998	0.053
10000	62832	0.999	0.015

The table demonstrates the following characteristics of this system:

- At $\Omega = 0$, $|H(j\Omega)| = 0$, indicating that the network does not pass a dc voltage. As the angular frequency Ω increases the magnitude function increases, and approaches $|H(j\Omega)| = 1$ for high frequencies. This is a "high-pass" system.
- At low frequencies the phase shift $\angle H(j\Omega) \approx \pi/2$. as the frequency increases the phase shift $\angle H(j\Omega) \to 0$, but is always positive, indicating that this is a "phase-lead" system.

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5 The Frequency Response of First and Second-Order Systems

5.1 First-Order Systems

The first-order system with time constant τ and an input/output differential equation

$$\tau\frac{dy}{dt} + y = K_o u(t) \tag{25}$$

where K_o is a constant, has a transfer function

$$H(s) = \frac{K_o}{\tau s + 1}. \tag{26}$$

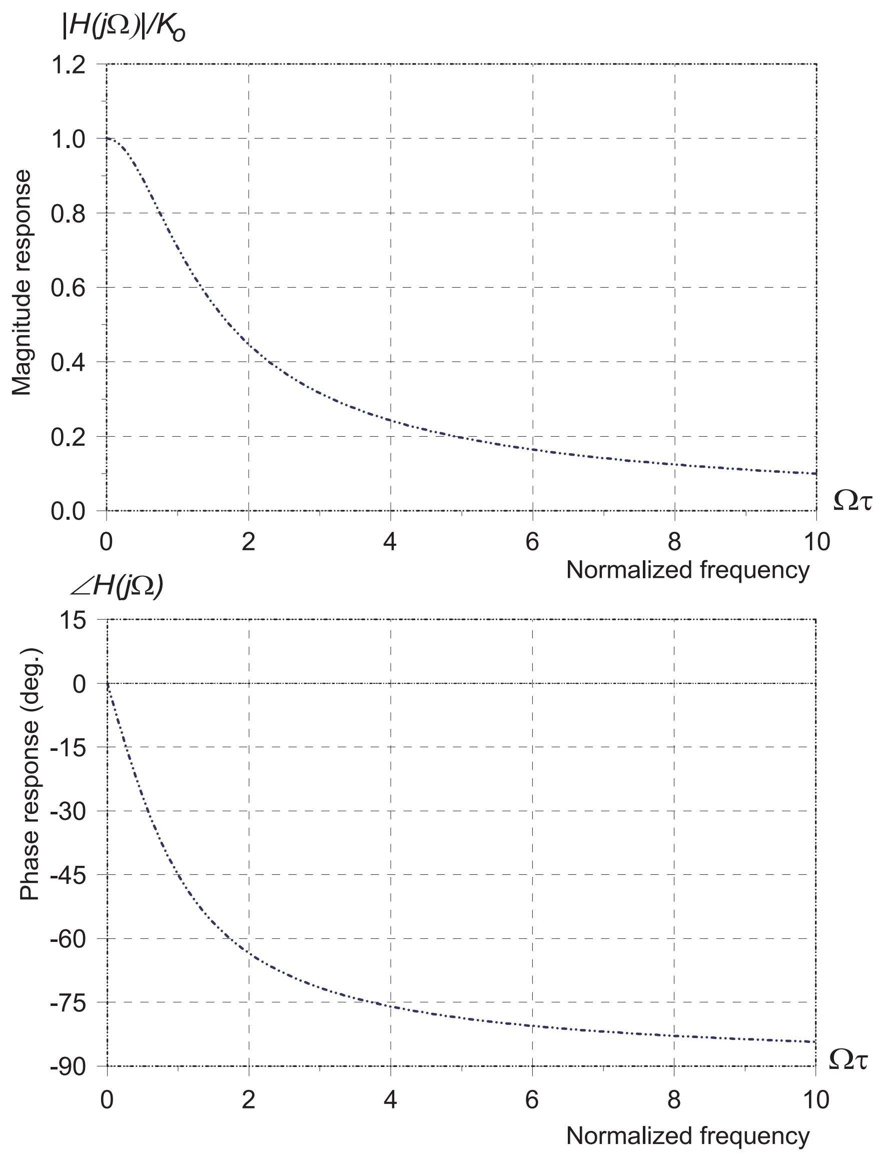

Figure 5: Magnitude and phase plots of the frequency response of a first-order system.

The frequency response function is found by direct substitution of $s = j\Omega$:

$$H(j\Omega) = \frac{K_o}{j\Omega\tau + 1}. \tag{27}$$

The magnitude and phase angle functions of the frequency response are

$$|H(j\Omega)| = \frac{K_o}{\sqrt{(\Omega\tau)^2 + 1}} \tag{28}$$

$$\phi(j\Omega) = \tan^{-1}(-\Omega\tau), \tag{29}$$

so that if an input $u(t) = \sin(\Omega t)$ is applied to the input, the steady-state response is

$$y_{ss}(t) = \frac{K_o}{\sqrt{(\Omega\tau)^2 + 1}} \sin\left(\Omega t + \phi(j\Omega)\right). \tag{30}$$

As the input frequency becomes small and approaches zero, the magnitude of $H(j\Omega)$ approaches K_o. As the angular frequency Ω increases the value of $|H(j\Omega)|$ approaches zero, indicating that the system *attenuates* high frequency sinusoidal inputs. A first-order system of this type is therefore a *low-pass system.*

The phase response $\phi(j\Omega)$ has a *lag* characteristic, because $\phi(j\Omega) < 0$ for all frequencies. At low frequencies ($\Omega \ll 1/\tau$) the phase shift is approximately zero, at a frequency $\Omega = 1/\tau$ the phase shift is $-\pi/4$ radians ($-45°$), and at high frequencies the phase shift approaches a maximum value of $-\pi/2$ radians ($-90°$).

The normalized magnitude, $|H(j\Omega)|/K_o$, and phase functions are shown against a normalized frequency scale ($\Omega\tau$) in Fig. 5.

■ Example 2

The height of fluid in a mixing tank is monitored by an electrical pressure transducer that generates a voltage proportional to the fluid pressure at the bottom of the tank. The transducer output is viewed on an oscilloscope, as shown in Fig. 6. The normal pressure variations in the tank are in the frequency range of 0–3 Hz. When the system was installed it was found that the measured output was contaminated by a strong 60 Hz noise signal, caused by electromagnetic radiation from nearby equipment. It was suggested that a first-order electrical low-pass filter, shown in Fig. 6, inserted between the transducer pre-amplifier and the oscilloscope would reduce the effects of the 60 Hz interference. The circuit values for the R-C circuit must be selected to attenuate the 60 Hz noise while not significantly affecting the pressure signals in the 0–3 Hz range. The specifications require that any sinusoidal signal with a frequency in the 0–3 Hz range be attenuated by no more than a factor of 0.9.

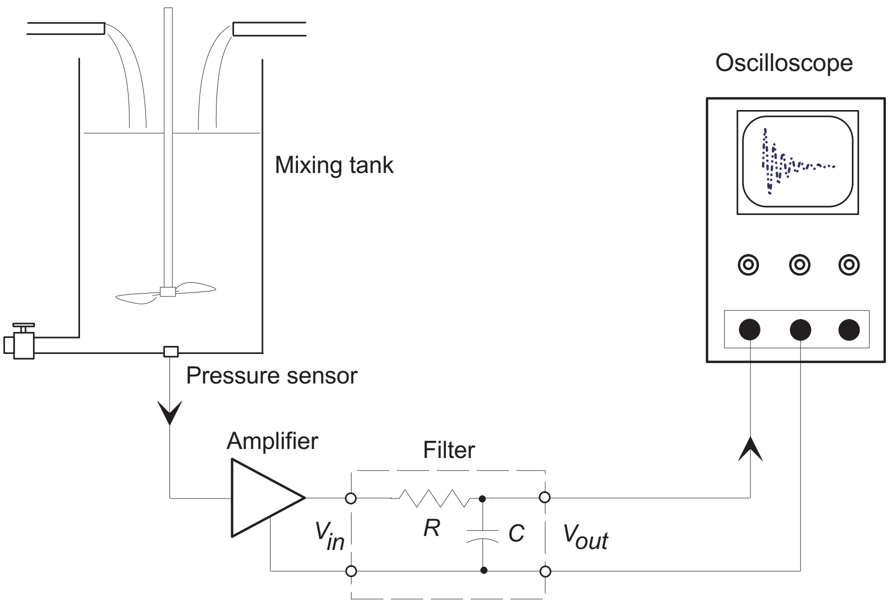

Figure 6: An electrical filter inserted in a measurement system to attenuate high frequency interference.

Solution: The transfer function for the filter as shown is

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{RCs + 1} \tag{i}$$

and the magnitude of the frequency response is (Eq. 29):

$$|H(j\Omega)| = \frac{|V_{out}|}{|V_{in}|} = \frac{1}{\sqrt{(\Omega\tau)^2 + 1}} \tag{ii}$$

where $\tau = RC$ is the system time constant. The filter design requires choosing an appropriate time constant τ, and then the selection of values for R and C to meet the time constant.

The response of a first-order system is a monotonically decreasing function of Ω. The time constant can be selected directly from Eq. (ii) by setting the magnitude to 0.9 at the specified frequency of 3 Hz, or $\Omega = 6\pi$ rad/s:

$$0.9 = \frac{1}{\sqrt{(\tau 6\pi)^2 + 1}}$$

which gives a value of $\tau = 0.026$s. With this value the response to all sinusoids below 3 Hz is in the range $0.9 \leq |H(j\Omega)| \leq 1.0$. The magnitude response function is shown in Fig. 7.

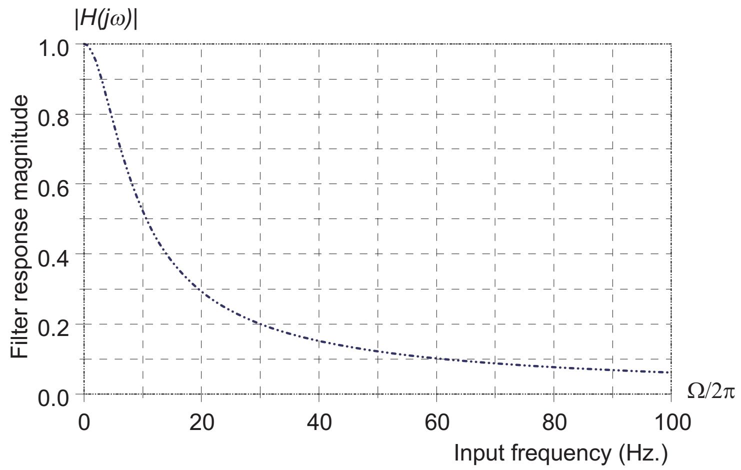

Figure 7: The frequency response magnitude of the electrical filter with a time constant of 0.026 seconds.

With this value for τ the response of the filter to a 60 Hz noise signal, $V_{in}\sin(120\pi t)$, is

$$\frac{V_{out}}{V_{in}} = \frac{1}{\sqrt{(0.0257 \times 120\pi)^2 + 1}} = 0.093$$

so that the noise amplitude at the filter output is reduced to less than 10% of its input value.

The values of R and C must then be selected so that $RC = 0.026$ s. Many combinations may be selected, for example $C = 0.2$ μfd, and $R = 130$ kohm. Practical considerations such as the input impedance of the oscilloscope and the output impedance of the transducer dictate the final choice of component values.

5.2 Second-Order Systems

Consider the second-order system with an input/output differential equation

$$\frac{d^2y}{dt^2} + 2\zeta\Omega_n\frac{dy}{dt} + \Omega_n^2 y = K_o u(t) \tag{31}$$

where Ω_n is the undamped natural frequency and ζ is the damping ratio, and K_o is a constant. The system transfer function is

$$H(s) = \frac{K_o}{s^2 + 2\zeta\Omega_n s + \Omega_n^2}, \tag{32}$$

and by substitution of $s = j\Omega$ the frequency response is

$$\begin{aligned} H(j\Omega) &= \frac{K_o}{(\Omega_n^2 - \Omega^2) + j2\zeta\Omega_n\Omega} \\ &= \frac{K_o/\Omega_n^2}{(1 - (\Omega/\Omega_n)^2) + j\,(2\zeta\Omega/\Omega_n)}. \end{aligned} \tag{33}$$

The magnitude and phase functions of the frequency response $H(j\Omega)$ are

$$|H(j\Omega)| = \frac{K_o/\Omega_n^2}{\sqrt{(1 - (\Omega/\Omega_n)^2)^2 + (2\zeta(\Omega/\Omega_n))^2}} \tag{34}$$

$$\phi(j\Omega) = \tan^{-1}\frac{-2\zeta(\Omega/\Omega_n)}{1 - (\Omega/\Omega_n)^2}. \tag{35}$$

It is convenient to plot the magnitude response in a normalized form by dividing by the factor K_o/Ω_n^2, and to define a normalized frequency scale Ω/Ω_n. Figure 8 shows the normalized magnitude and phase plots for this second-order system with the damping ratio ζ as a parameter. At low frequencies the normalized magnitude response is approximately unity, and for values of $\zeta > 1$ (that is when the system is over-damped) the gain function shows a monotonically decreasing value with frequency. For lightly damped systems ($\zeta \ll 1$) however, the gain function has a *resonant peak* in the region of Ω_n before decaying to zero at higher frequencies. In the region of this resonance the system amplifies the input, that is the amplitude of the output sinusoid is greater than that of the input. As the value of ζ approaches zero the resonant peak becomes narrower and the peak amplitude increases.

The resonance associated with a lightly damped system may be used to advantage in some applications, particularly electronic communications systems where it is often important to design a filter to emphasize sinusoidal components in a particular frequency band. In many other applications, such as mechanical systems, unwanted resonances can result in large amplitude vibrations that must be prevented by careful design, or by the addition of specialized vibration suppressors.

The maximum value of the magnitude function occurs at a frequency that is dependent on the damping ratio ζ. This frequency may be found by differentiating Eq. (35) with respect to Ω, equating the result to zero, and solving to find the frequency Ω_m of the maximum response, giving

$$\Omega_m = \Omega_n\sqrt{1 - 2\zeta^2} \tag{36}$$

for $\zeta \leq 1/\sqrt{2}$. The resonance peak occurs at a frequency less than the undamped natural frequency Ω_n, with the shift increasing towards zero as the damping is increased. Substitution of Ω_m into Eq. (35) gives the value of the amplification at resonance as a function of the damping ratio:

$$|H(j\Omega_m)| = \frac{K_0/\Omega_n^2}{2\zeta\sqrt{1 - \zeta^2}}. \tag{37}$$

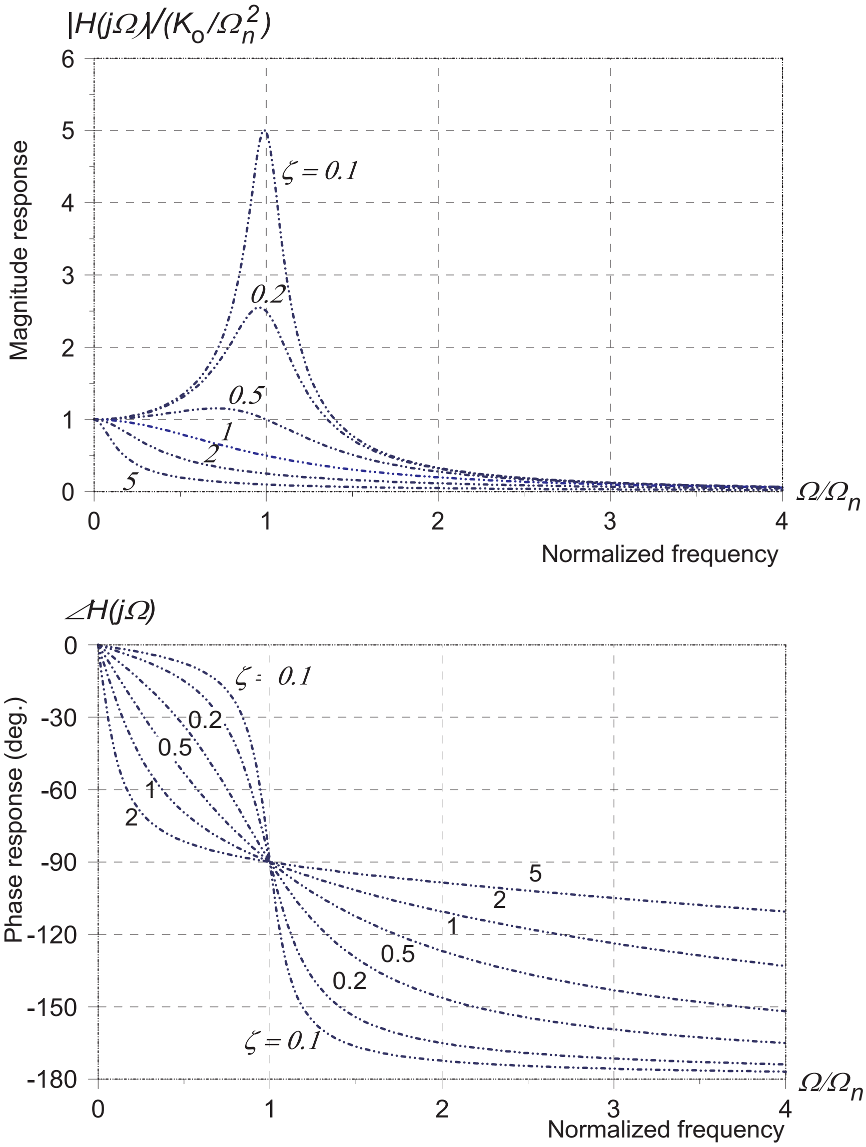

Figure 8: Magnitude and phase frequency response plots of a second-order system

Figure 9 shows the amplification factor and the frequency of the maximum response for underdamped second-order systems. For values of $\zeta \geq 0.707$ no peak occurs in the response, and the magnitude monotonically decreases with increasing frequency.

The phase response of the second-order system has a phase lag characteristic that approaches zero as the input frequency approaches zero, and tends towards a maximum phase shift of $-\pi$ radians ($-180°$) as the frequency becomes large. At a frequency of $\Omega = \Omega_n$ the phase shift is exactly $-\pi/2$ radians ($-90°$). The slope and width of the transition region depend on the value of the damping ratio ζ.

■ Example 3

A RLC filter, used in a communication system, is shown in Fig. 10. Find the frequency

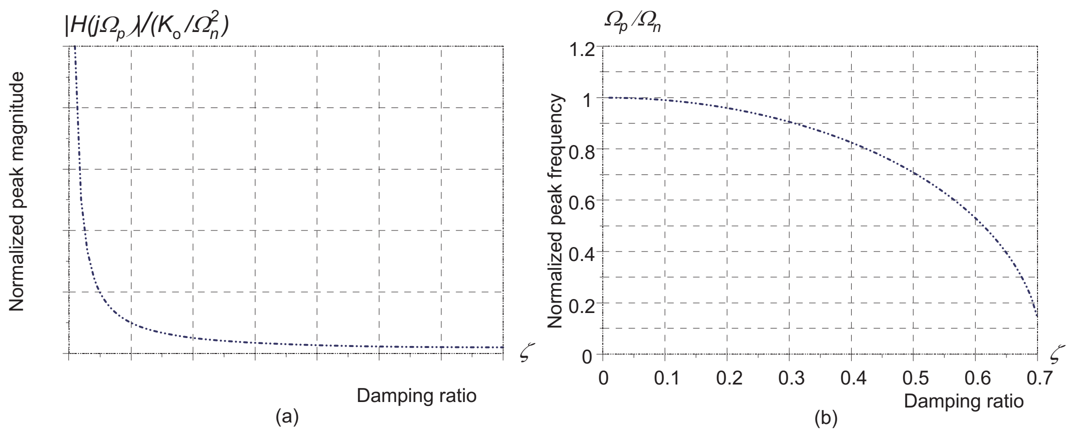

Figure 9: The resonant peak of an underdamped second-order system frequency response; (a) the magnitude of the peak response, and (b) the frequency of the peak.

at which the response has a maximum amplitude, and the value of the magnitude response at that frequency.

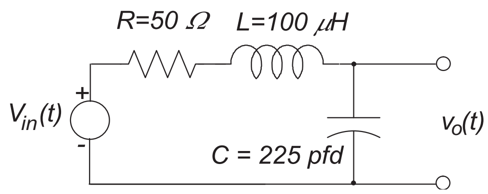

Figure 10: A series RLC filter.

Solution: The filter has a second-order transfer function

$$H(s) = \frac{V_o(s)}{V_{in}(s)} = \frac{1}{LCs^2 + RLs + 1} \tag{i}$$

With the given values ($R = 50\ \Omega$, $L = 100\ \mu$H, and $C = 225$ pfd) the system's undamped natural frequency Ω_n and damping ratio ζ are:

$$\Omega_n = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(100 \times 10^{-6}) \times (225 \times 10^{-12})}} = 6.667 \times 10^6 \text{ rad/s} \tag{ii}$$

$$\zeta = \frac{R}{2\sqrt{L/C}} = \frac{50}{2\sqrt{100 \times 10^{-6}/(225 \times 10^{-12})}} = 0.0375 \tag{iii}$$

The peak resonance frequency is given by Eq. (36):

$$\Omega_m = \Omega_n\sqrt{1 - 2\zeta^2} = 6.667 \times 10^6 \times \sqrt{1 - 2 \times 0.0375^2} = 6.657\text{rad/sec} \tag{iv}$$

or 1.059 MHz. At Ω_m the peak amplification ratio of the filter is given by Eq. (37) as:

$$|H(j\Omega_m)| = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} = 13.34 \tag{v}$$

which indicates that the output voltage is significantly larger than the input at this frequency.

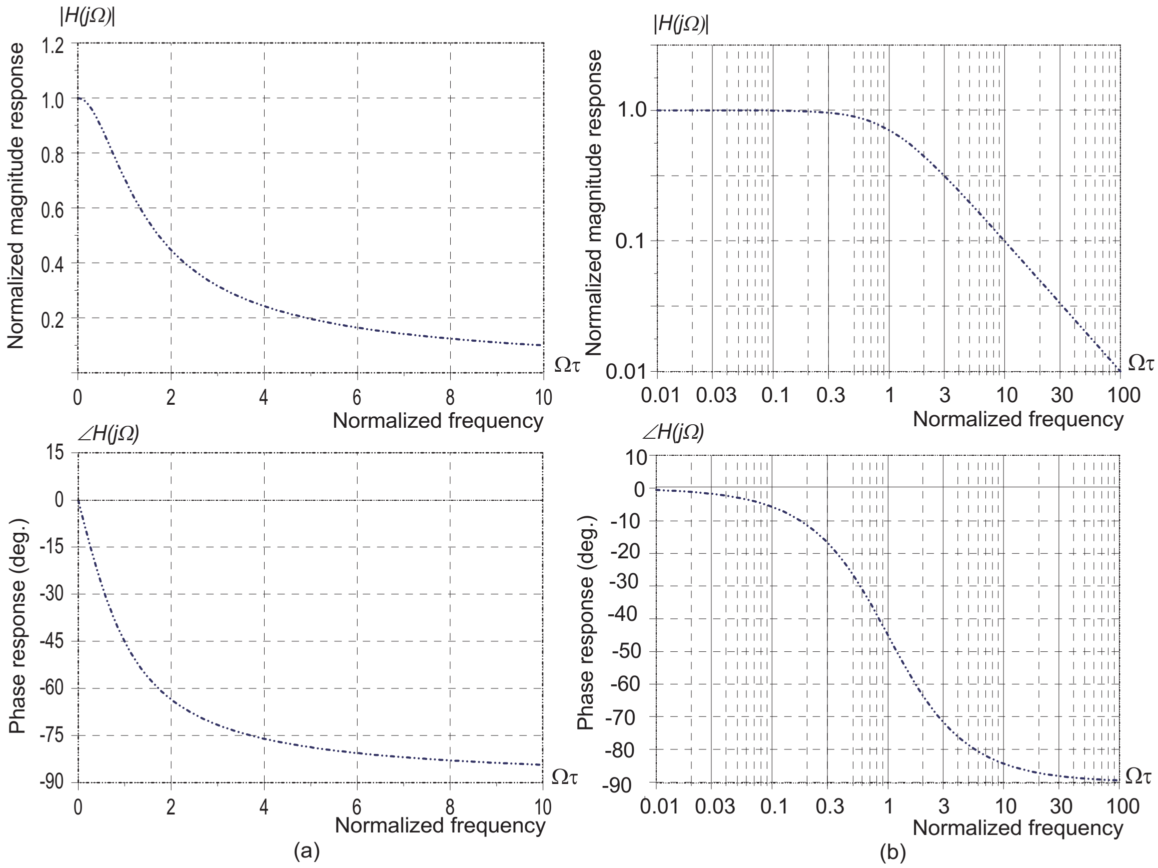

Figure 11: Comparison of magnitude and phase characteristics of a first-order system $H(s) = 1/(\tau s + 1)$, drawn (a) using linear scales and (b) as Bode plots using the logarithmic scales.

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6 Logarithmic (Bode) Frequency Response Plots

Frequency response characteristics are almost always plotted using logarithmic scales. In particular, the magnitude function $|H(j\Omega)|$ is plotted against frequency on a log-log scale, and the phase $\angle H(j\Omega)$ is plotted on a linear-log scale. For example, in Fig. 11 the frequency response functions of a typical first-order system $\tau dy/dt + y = u(t)$, similar to that discussed in Section (5.1), is plotted on (a) linear axes, and (b) logarithmically scaled axes. It can be seen that while two sets of plots convey the same information, they have a different appearance. The logarithmic frequency scale has the effect of expanding the low frequency region of the plots while compressing the high frequencies. The logarithmic magnitude plot can be seen to exhibit straight line asymptotic behavior at high and low frequencies. Similarly, Fig. 12 compares the linear and logarithmic versions of the second-order system described in Section 5.2. In this case the curves are a function of the damping ratio ζ, but all curves on the magnitude plots can be seen to approach a pair of straight line asymptotes as the frequency becomes very small, and as the frequency becomes very large.

In the 1940's H. W. Bode introduced the logarithmic frequency response plots as a simplified method for sketching approximate frequency response characteristics of electronic feedback ampli-

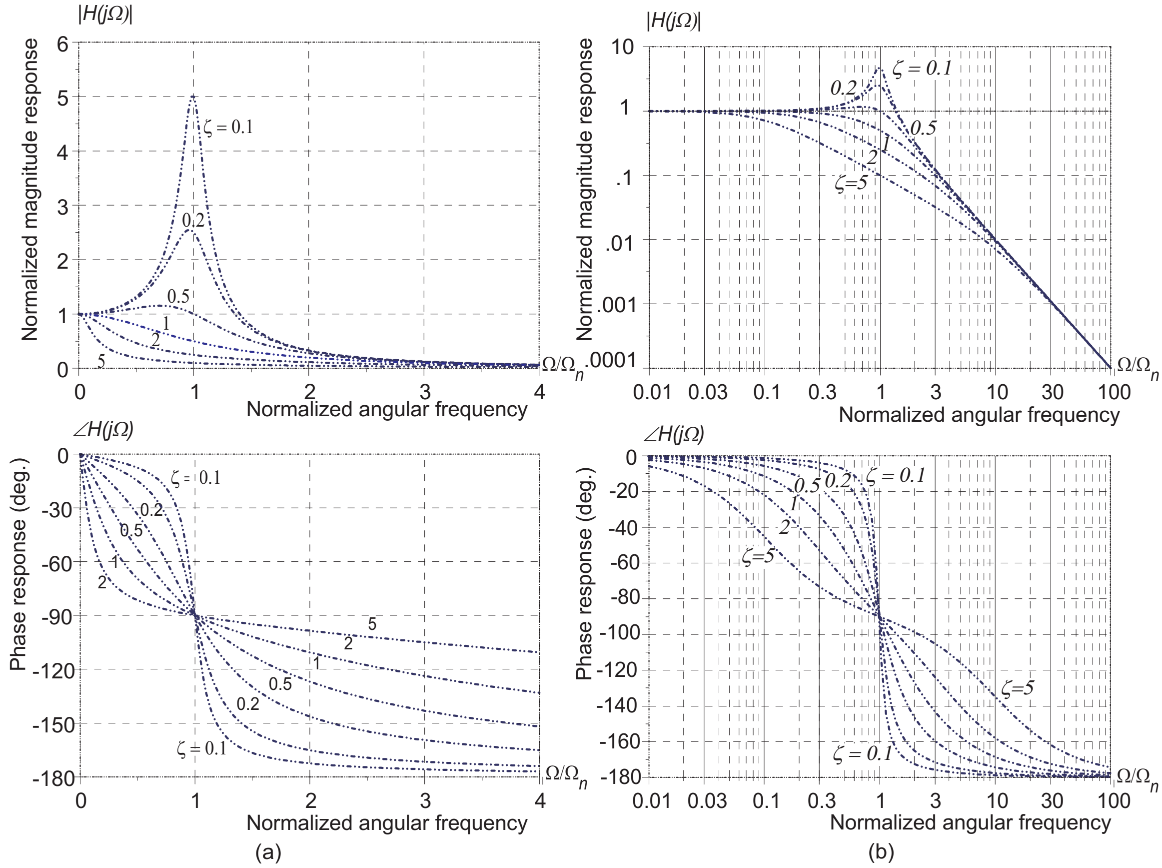

Figure 12: Comparison of magnitude and phase characteristics of a second-order system $H(s) = \Omega_n^2/(s^2 + 2\zeta\Omega_n s + \Omega_n^2)$, drawn (a) using linear scales and (b) as Bode plots using the logarithmic scales.

fiers. Bode plots, named after him, have subsequently been widely used in linear system design and analysis, and in feedback control system design and analysis[5–8]. The Bode sketching method provides an effective means of approximating the frequency response of a complex system by combining of the responses of simple first and second-order systems.

6.1 Logarithmic Amplitude and Frequency Scales:

6.1.1 Logarithmic Amplitude Scale: The Decibel

Bode magnitude plots are frequently plotted using the decibel logarithmic scale to display the function $|H(j\Omega)|$. The Bel, named after Alexander Graham Bell, is defined as the logarithm to base 10 of the *ratio* of two *power* levels. In practice the Bel is too large a unit, and the decibel (abbreviated dB), defined to be one tenth of a Bel, has become the standard unit of logarithmic power ratio. The power flow $\mathcal{P}$ into any element in a system, may be expressed in terms of a logarithmic ratio Q to a *reference* power level $\mathcal{P}_{ref}$:

$$Q = \log_{10}\left(\frac{\mathcal{P}}{\mathcal{P}_{ref}}\right) \text{ Bel} \quad \text{or} \quad Q = 10\log_{10}\left(\frac{\mathcal{P}}{\mathcal{P}_{ref}}\right) \text{ dB}. \tag{38}$$

Decibels	Power Ratio	Amplitude Ratio
-40	0.0001	0.01
-20	0.01	0.1
-10	0.1	0.3162
-6	0.25	0.5
-3	0.5	0.7071
0	1.0	1.0
3	2.0	1.414
6	4.0	2.0
10	10.0	3.162
20	100.0	10.0
40	10000.0	100.0

Table 1: Common Decibel quantities and their corresponding power and amplitude ratios.

Because the power dissipated in a D–type element is proportional to the square of the amplitude of a system variable applied to it, when the ratio of across or through variables is computed the definition becomes

$$Q = 10\log_{10}\left(\frac{A}{A_{ref}}\right)^2 = 20\log_{10}\left(\frac{A}{A_{ref}}\right) \text{ dB.} \tag{39}$$

where A and A_{ref} are amplitudes of variables. [2] Table 1 expresses some commonly used decibel values in terms of the power and amplitude ratios.

The magnitude of the frequency response function $|H(j\Omega)|$ is defined as the *ratio* of the amplitude of a sinusoidal output variable to the amplitude of a sinusoidal input variable. This ratio is expressed in decibels, that is

$$20\log_{10}|H(j\Omega)| = 20\log_{10}\frac{|Y(j\Omega)|}{|U(j\Omega)|} \text{ dB.}$$

As noted this usage is not strictly correct because the frequency response function does not define a power ratio, and the decibel is a dimensionless unit whereas $|H(j\Omega)|$ may have physical units.

6.1.2 Logarithmic Frequency Scales

In the Bode plots the frequency axis is plotted on a logarithmic scale. Two logarithmic units of frequency ratio are commonly used: the *octave* which is defined to be a frequency ratio of 2:1, and the *decade* which is a ratio of 10:1. Given two frequencies Ω_1 and Ω_2 the frequency ratio $W = (\Omega_1/\Omega_2)$ between them may be expressed logarithmically in units of decades or octaves by the relationships

$$\begin{aligned} W &= \log_2(\Omega_1/\Omega_2) \text{ octaves} \\ &= \log_{10}(\Omega_1/\Omega_2) \text{ decades.} \end{aligned}$$

[2] This definition is only strictly correct when the two amplitude quantities are measured across a common D–type (dissipative) element. Through common usage, however, the decibel has been effectively redefined to be simply a convenient logarithmic measure of amplitude ratio of any two variables. This practice is widespread in texts and references on system dynamics and control system theory. In this book we have also adopted this convention.

The terms "above" and "below" are commonly used to express the positive and negative values of logarithmic values of W. A frequency of 100 rad/s is said to be two octaves (a factor of 2^2) above 25 rad/s, while it is three decades (a factor of 10^{-3}) below 100,000 rad/s.

6.2 Asymptotic Bode Plots of Low-Order Transfer Functions

The Bode plots consist of (1) a plot of the logarithmic magnitude (gain) function, and (2) a separate linear plot of the phase shift, both plotted on a logarithmic frequency scale. In this section we develop the plots for first and second-order terms in the transfer function. The approximate sketching methods described here are based on the fact that an approximate log–log magnitude plot can be derived from a set of simple straight line asymptotic plots that can be easily combined graphically.

The system transfer function in terms of factored numerator and denominator polynomials is

$$H(s) = K\frac{(s-z_1)(s-z_2)\dots(s-z_{m-1})(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_{n-1})(s-p_n)}, \tag{40}$$

where the z_i, for $i = 1, \dots, m$, are the system zeros, and the p_i, for $i = 1, \dots, n$, are the system poles.

In general a system may have complex conjugate pole and zero pairs, real poles and zeros, and possibly poles or zeros at the origin of the s-plane. Bode plots are constructed from a rearranged form of Eq. (40), in which complex conjugate poles and zeros are combined into second-order terms with real coefficients. For example a pair of complex conjugate poles $s_i, s_{i+1} = \sigma_i \pm j\Omega_i$ is written

$$\left.\frac{1}{(s-(\sigma_i + j\Omega_i))\,(s-(\sigma_i - j\Omega_i))}\right|_{s=j\Omega} = \left(\frac{1}{\Omega_n^2}\right)\frac{1}{(1-(\Omega/\Omega_n)^2)+j2\zeta\Omega/\Omega_n} \tag{41}$$

and described by parameters Ω_n and ζ. The constant terms $1/\Omega_n^2$ is absorbed into a redefinition of the gain constant K.

In the following sections Bode plots are developed for the first and second-order numerator and denominator terms:

6.2.1 Constant Gain Term:

The simplest transfer function is a constant gain, that is $H(s) = K$.

$$|H(j\Omega)| = K \quad \text{and} \quad \angle H(j\Omega) = 0, \tag{42}$$

and converting to the logarithmic decibel scale

$$20\log_{10}|H(j\Omega)| = 20\log_{10} K \quad \text{and} \quad \angle H(j\Omega) = 0 \text{ dB}. \tag{43}$$

The Bode magnitude plot is a horizontal line at the appropriate gain and the phase plot is identically zero for all frequencies.

6.2.2 A Pole at the Origin of the s-plane:

A single pole at the origin of the s-plane, that is $H(s) = 1/s$, has a frequency response

$$|H(j\Omega)| = \frac{1}{\Omega} \quad \text{and} \quad \angle H(j\Omega) = -\pi/2. \tag{44}$$

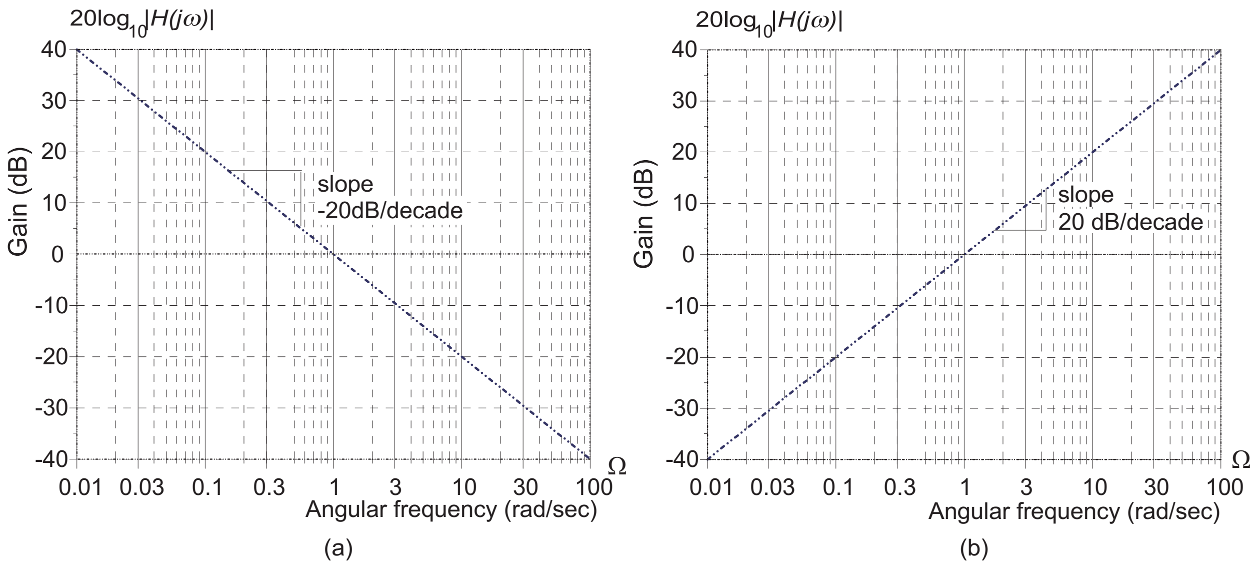

Figure 13: Bode magnitude plots for (a) a single pole at the origin of the s-plane, and (b) for a single zero at the origin. The phase plots are not shown, they are a constant of $-\pi/2$ for the pole, and $\pi/2$ for the zero.

The value of the magnitude function in logarithmic units is

$$\log|H(j\Omega)| = -\log(\Omega) \tag{45}$$

or using the decibel scale

$$20\log_{10}|H(j\Omega)| = -20\log_{10}(\Omega) \text{ dB}. \tag{46}$$

The decibel based Bode magnitude plot is therefore a straight line with a slope of -20 dB/decade and passing through the 0 dB line ($|H(j\Omega)| = 1$) at a frequency of 1 rad/s. The phase plot is a constant value of $-\pi/2$ rad, or $-90°$, at all frequencies. The magnitude Bode plot for this system is shown in Fig. 13a.

6.2.3 A Single Zero at the Origin:

A single zero at the origin of the s-plane, that is $H(s) = s$, has a frequency response $H(j\Omega)$ with magnitude and phase

$$|H(j\Omega)| = \Omega \quad \text{and} \quad \angle H(j\Omega) = \pi/2. \tag{47}$$

The logarithmic magnitude function is therefore

$$\log|H(j\Omega)| = \log(\Omega) \tag{48}$$

or in decibels

$$20\log_{10}|H(j\Omega)| = 20\log_{10}(\Omega) \text{ dB}. \tag{49}$$

The Bode magnitude plot is a straight line with a slope of +20 dB/decade. This curve also has a gain of 0 dB (unity gain) at a frequency of 1 rad/s. The phase plot is a constant of $\pi/2$ radians, or $+90°$, at all frequencies. Figure 13b shows the magnitude Bode plot for this term.

6.2.4 A Single Real Pole

The frequency response of a single real pole factor written in the form

$$H(s) = \frac{1}{\tau s + 1} \tag{50}$$

was derived in Section 5.1:

$$|H(j\Omega)| = \frac{1}{\sqrt{(\Omega\tau)^2 + 1}} \quad \text{and} \quad \angle H(j\Omega) = \tan^{-1}(-\Omega\tau). \tag{51}$$

The logarithmic magnitude function is

$$\log|H(j\Omega)| = -0.5\log\left((\Omega\tau)^2 + 1\right), \tag{52}$$

or as a decibel function

$$20\log_{10}|H(j\Omega)| = -10\log_{10}\left((\Omega\tau)^2 + 1\right) \text{ dB}. \tag{53}$$

When $\Omega\tau \ll 1$, the first term may be ignored and the magnitude may be approximated by a *low-frequency asymptote*

$$\lim_{\Omega\tau\to 0} 20\log_{10}|H(j\Omega)| = -10\log_{10}(1) = 0 \text{ dB} \tag{54}$$

which is a horizontal line on the plot at 0dB (unity) gain. At high frequencies, for which $\Omega\tau \gg 1$, the unity term in the magnitude expression Eq. (53) may be ignored the magnitude function is approximated by a *high-frequency asymptote*

$$20\log_{10}|H(j\Omega)| \approx -10\log_{10}((\Omega\tau)^2) = -20\log_{10}(\Omega) - 20\log_{10}(\tau) \quad \text{dB}. \tag{55}$$

which is a straight line when plotted against $\log(\Omega)$, with a slope of -20 dB/decade. The high and low frequency asymptotes intersect on the plot on the 0 dB line at a *corner* or *break* frequency of $\Omega = 1/\tau$. The complete asymptotic Bode magnitude plot as defined by these two line segments is shown in Fig. 14a using a normalized frequency axis. The exact response is also shown in the figure; at the break frequency $\Omega = 1/\tau$ the actual response is $20\log_{10}|H(j\Omega)| = -10\log_{10}(2) = -3$ dB.

The phase characteristic is also plotted against a normalized frequency scale in Fig. 14a. At low frequencies the phase shift approaches 0 radians. It passes through a phase shift of $-\pi/4$ radians at the break frequency $\Omega = 1/\tau$, and asymptotically approaches a maximum phase lag of $-\pi/2$ radians as the frequency becomes very large. A piece-wise linear approximation may be made by assuming that the curve has a phase shift of 0 radians at frequencies more than one decade below the break frequency, a phase shift of $-\pi/2$ radians at frequencies more than a decade above the break frequency, and a linear transition in phase between these two frequencies on the logarithmic frequency scale. This approximation is within 0.1 radians of the exact value at all frequencies.

6.2.5 A Single Real Zero

A numerator term, corresponding to a single real zero, written in the form $H(s) = \tau s + 1$ (where τ is not strictly a time constant), is handled in a manner similar to a real pole. In this case

$$H(j\Omega) = j\Omega\tau + 1$$

and the magnitude and phase responses are

$$|H(j\Omega)| = \sqrt{1+(\Omega\tau)^2} \quad \text{and} \quad \angle H(j\Omega) = \tan^{-1}(\Omega\tau) \tag{56}$$

respectively. In decibels the magnitude expression is

$$20\log_{10}|H(j\Omega)| = 10\log_{10}(1+(\Omega\tau)^2) \text{ dB}. \tag{57}$$

The low frequency asymptote is found by assuming that $\Omega\tau \ll 1$ in which case

$$\lim_{\Omega\tau\to 0} 20\log_{10}|H(j\Omega)| = 10\log_{10}(1) = 0 \text{ dB}, \tag{58}$$

and the high frequency asymptote is found by assuming that $\Omega\tau \gg 1$,

$$20\log_{10}|H(j\Omega)| \approx 20\log_{10}(\Omega\tau) = 20\log_{10}(\Omega) - 20\log_{10}(\tau) \text{ dB} \quad \text{when } \Omega \gg 1/\tau \tag{59}$$

which is a straight line on the log-log plot, with a slope of +20 dB/decade. The break frequency, defined by the intersection of these two asymptotes is at a frequency $\Omega = 1/\tau$, and at this frequency the exact value of $|H(j\Omega)|$ is $\sqrt{2}$ or +3 dB. The complete asymptotic Bode magnitude plot using a normalized frequency scale is shown in Fig. 14b.

The phase characteristic asymptotically approaches 0 radians at low frequencies and approaches a maximum phase lead of $\pi/2$ radians at frequencies much greater than the break frequency. At the break frequency the phase shift is $\pi/4$ radians. A piece-wise linear approximation, similar to that described for a real pole, is also shown in the Fig. 14b.

6.2.6 Complex Conjugate Pole Pair:

The classical second-order system,

$$H(s) = \frac{\Omega_n^2}{s^2 + 2\zeta\Omega_n s + \Omega_n^2} \tag{60}$$

described in Section 5.2, has a frequency response

$$|H(j\Omega)| = \frac{1}{\sqrt{(1-(\Omega/\Omega_n)^2)^2 + (2\zeta(\Omega/\Omega_n))^2}} \tag{61}$$

$$\text{and} \quad \angle H(j\Omega) = \tan^{-1}\frac{-2\zeta\,(\Omega/\Omega_n)}{\left(1-(\Omega/\Omega_n)^2\right)}. \tag{62}$$

In logarithmic units the magnitude response is

$$20\log_{10}|H(j\Omega)| = -10\log_{10}\left[\left(1-(\Omega/\Omega_n)^2\right)^2 + (2\zeta(\Omega/\Omega_n))^2\right] \tag{63}$$

The Bode forms of the magnitude and phase responses are plotted in Fig. 15a, with the damping ratio ζ as a parameter. The low-frequency asymptote is found by assuming that $\Omega/\Omega_n \ll 1$ so that

$$\lim_{(\Omega/\Omega_n)\to 0}(20\log_{10}|H(j\Omega)|) = -10\log_{10}(1) = 0 \text{ dB}. \tag{64}$$

The high frequency response can be found by retaining only the dominant term when $\Omega/\Omega_n \gg 1$:

$$\begin{aligned} 20\log_{10}|H(j\Omega)| &\approx -10\log_{10}\left[(\Omega/\Omega_n)^4\right] \\ &= -40\log_{10}(\Omega) + 40\log_{10}(\Omega_n) \text{ dB} \quad \text{when } \Omega \gg \Omega_n, \end{aligned} \tag{65}$$

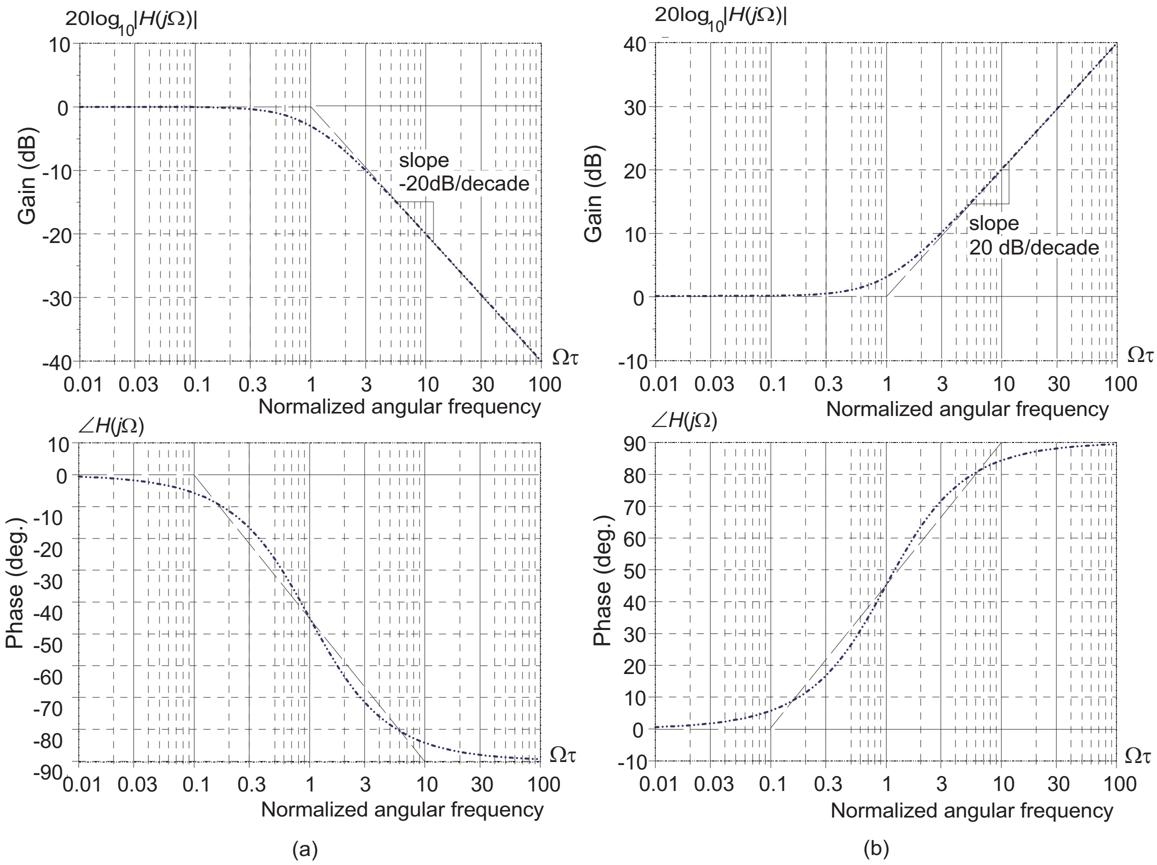

Figure 14: Bode magnitude and phase plots for (a) a single real pole, and (b) for a single real zero.

which is a linear function of $\log_{10} \Omega$ with a slope of -40 dB/decade. The two asymptotes intersect at a break frequency of $\Omega = \Omega_n$ as shown in Fig. 15a. The straight line asymptotic form does not account in any way for the damping ratio. The resonance peak (for values of $\zeta < 0.707$) must be sketched in after the asymptotes have been drawn. Figure 16, which contains the same data as Fig. 9, plots the logarithmic magnitude correction and frequency of the resonant peak as a function of ζ from Eqs. (36) and (37); it is a simple matter to sketch in the resonant peak from these values.

The phase characteristic asymptotically approaches 0 radians at low frequencies, has a phase lag of $-\pi/2$ at the break frequency Ω_n, and approaches $-\pi$ radians at high frequencies. The steepness of the transition is a function of the damping ratio ζ and so must be sketched using the information contained in Fig. 15a.

6.2.7 Complex Conjugate Zero Pair

Bode plots for a pair of complex conjugate zeros can be derived in a manner similar to the conjugate pole pair described above. In this case the block is assumed to have a transfer function

$$H(s) = \frac{1}{\Omega_n^2}\left(s^2 + 2\zeta\Omega_n s + \Omega_n^2\right) \tag{66}$$

and a frequency response

$$|H(j\Omega)| = \sqrt{\left(1 - (\Omega/\Omega_n)^2\right)^2 + \left(2\zeta(\Omega/\Omega_n)\right)^2} \tag{67}$$

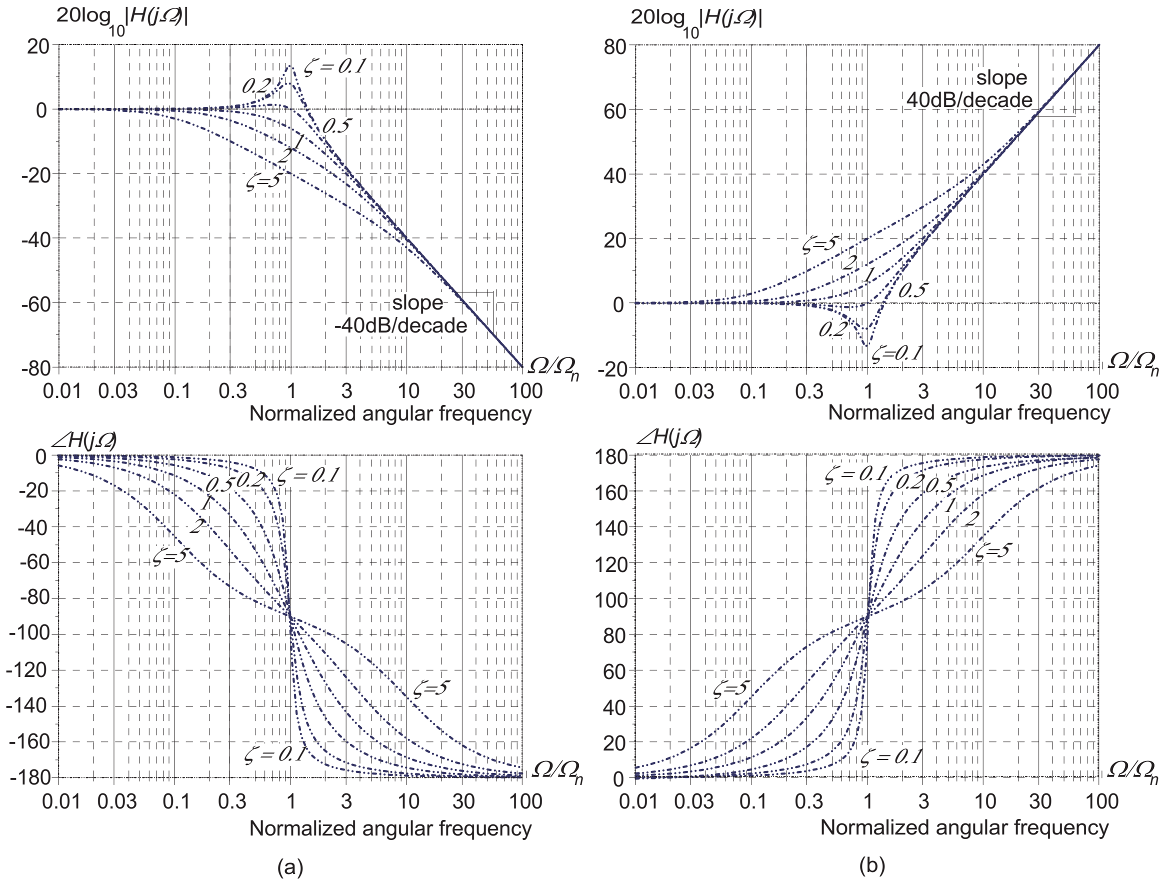

Figure 15: Bode magnitude and phase plots for (a) a complex conjugate pole pair, and (b) for a complex conjugate zero pair.

$$\text{and} \quad \angle H(j\Omega) \quad = \quad \tan^{-1} \frac{2\zeta\,(\Omega/\Omega_n)}{\left(1-(\Omega/\Omega_n)^2\right)}. \tag{68}$$

The logarithmic magnitude response is

$$20\log_{10}|H(j\Omega)| = 10\log_{10}\left[\left(1-(\Omega/\Omega_n)^2\right)^2 + (2\zeta(\Omega/\Omega_n))^2\right] \text{ dB} \tag{69}$$

The asymptotic responses are derived in a similar manner to the complex pole pair; the low frequency asymptote is

$$\lim_{(\Omega/\Omega_n)\to 0}(20\log_{10}|H(j\Omega)|) = 10\log_{10}(1) = 0 \text{ dB}, \tag{70}$$

and the high frequency asymptote is

$$\begin{aligned} 20\log_{10}|H(j\Omega)| &\approx 10\log_{10}\left[(\Omega/\Omega_n)^4\right] \\ &= 40\log_{10}(\Omega) - 40\log_{10}(\Omega_n) \text{ dB} \quad \text{for } \Omega \gg \Omega_n. \end{aligned} \tag{71}$$

The exact form of the magnitude response is plotted in Fig. 15b. This is effectively an inverse of the characteristic of Fig. 15a. There is a “notch” in the response in the region of the frequency Ω_n,

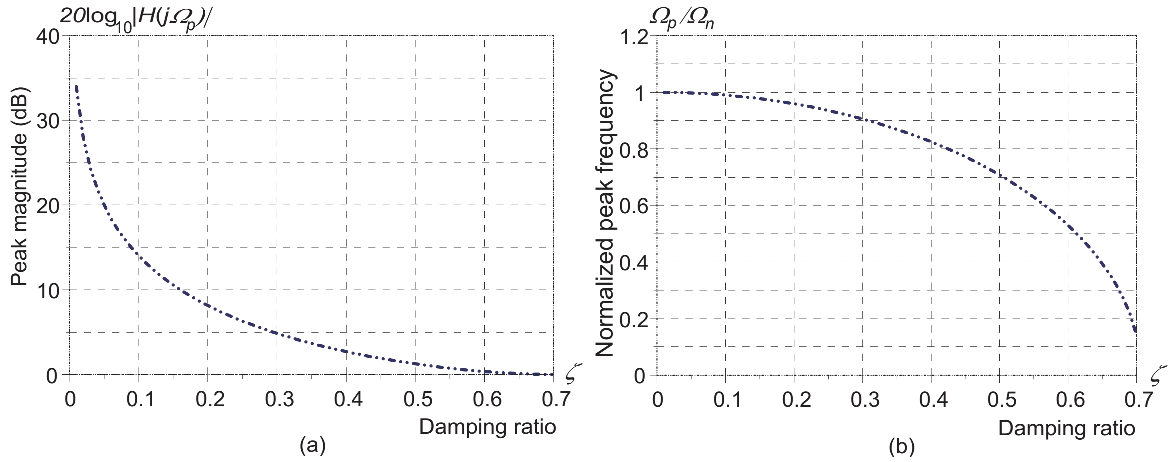

Figure 16: (a) Second-order resonant peak value in decibels, and (b) the frequency at which the peak occurs. These curves may be used to estimate corrections to the asymptotic Bode plots for lightly damped pole and zero pairs.

Description	Transfer Function	Break Frequency (radians/sec.)	High Frequency Slope (dB/decade)
Constant gain	K	-	0
Pole at the origin	$\frac{1}{s}$	-	-20
Zero at the origin	s	-	+20
Real pole	$\frac{1}{\tau s+1}$	$1/\tau$	-20
Real zero	$(\tau s+1)$	$1/\tau$	+20
Conjugate poles	$\frac{\Omega_n^2}{s^2+2\zeta\Omega_n s+\Omega_n^2}$	Ω_n	-40
Conjugate zeros	$\frac{1}{\Omega_n^2}(s^2+2\zeta\Omega_n s+\Omega_n^2)$	Ω_n	+40

Table 2: Summary of asymptotic magnitude Bode plot parameters for the seven basic blocks.

and the depth is a function of the parameter ζ. The plot has a low frequency asymptote of 0 dB, a break frequency of $\Omega = \Omega_n$, and a high-frequency asymptote is a straight line with a slope of +40 dB/decade. The phase characteristic is also a flipped version of that of a pair of complex conjugate poles; it approaches 0 radians at low frequencies, passes through $-\pi/2$ at the break frequency, and shows a maximum phase lead of π radians at high frequencies. As above, the slope of the curve in the transition region is dependent on the value of ζ.

6.2.8 Summary

The essential features of the asymptotic forms of the seven components of the magnitude plot are summarized in Table 2.

6.3 Bode Plots of Higher Order Systems

If a system with transfer function $H(s) = KN(s)/D(s)$ is expressed as a product of the terms in Table 2, that is

$$\begin{aligned} H(s) &= K\frac{N(s)}{D(s)} \\ &= K'(N_1(s)\ldots N_m(s)) \times (D_1(s)\ldots D_n(s)) \end{aligned} \tag{72}$$

where the factors $N_i(s)$ are first or second order zero terms, and the $D_i(s)$ are pole terms, and K' is a modified constant factor. For example

$$\begin{aligned} H(s) &= \frac{10(s+3)}{(s+0.5)(s+5)} = K'N_1(s)D_1(s)D_2(s) \\ &= 12 \times (\frac{1}{3}s+1) \times \frac{1}{2s+1} \times \frac{1}{0.2s+1}. \end{aligned}$$

When complex numbers are represented in polar form, the magnitude of a product is the product of the component magnitudes, and the angle of a product is the sum of the component angles, so that Eq. (72) may be expressed in terms of its magnitude and phase functions:

$$|H(j\Omega)| = K' \times |N_1(j\Omega)| \times \ldots \times |N_m(j\Omega)| \times \left|\frac{1}{D_1(j\Omega)}\right| \times \ldots \times \left|\frac{1}{D_n(j\Omega)}\right| \tag{73}$$

$$\angle H(j\Omega) = \angle N_1(j\Omega) + \ldots + \angle N_m(j\Omega) - \angle\frac{1}{D_1(j\Omega)} - \ldots - \angle\frac{1}{D_n(j\Omega)} \tag{74}$$

The logarithm of a product is the sum of the logarithms of its factors, so that Eq. (73) may be written

$$\log|H(j\Omega)| = \log K' + \log|N_1(j\Omega)| + \ldots + \log|N_m(j\Omega)| + \log\left|\frac{1}{D_1(j\Omega)}\right| + \ldots + \log\left|\frac{1}{D_n(j\Omega)}\right| \tag{75}$$

Equations (75) and (74) express the overall magnitude and phase responses as a *sum of component responses* of first and second-order elementary "building blocks". In practice Bode plots are constructed by graphically adding the individual response components. Given the transfer function $H(s)$ of a linear system, the following steps are used to construct the Bode magnitude plot:

1. Factor the numerator and denominator of the transfer function into the constant, first-order and quadratic terms in the form described in the previous section.

2. Identify the break frequency associated with each factor.

3. Plot the asymptotic form of each of the factors on log–log axes.

4. Graphically add the component asymptotic plots to form the overall plot in straight line form.

5. "Round out" the corners in the straight line approximate curve by hand, using the known values of the responses at the break frequencies (±3dB for first-order sections, and dependent upon ζ for quadratic factors).

The phase plot is constructed by graphically by adding the component phase responses. The individual plots are drawn, either as the piece-wise linear approximation for the first-order poles, or in a smooth form from the exact plot, and then these are added to find the total phase shift at any frequency.

■ **Example 4**

Plot the Bode magnitude and phase plots of a third-order system described by the transfer function

$$H(s) = \frac{40s + 4}{s^3 + 2s^2 + 2s}$$

Solution: The transfer function is rewritten

$$\begin{aligned} H(s) &= \frac{4(10s+1)}{s\,(s^2+2s+2)} \\ &= 2\,(10s+1)\left(\frac{1}{s}\right)\left(\frac{2}{s^2+2s+2}\right) \end{aligned} \qquad \text{(i)}$$

indicating four component terms:

1. A constant gain term of $H_1(s) = 2$,
2. A single real pole at the origin $H_2(s) = 1/s$,
3. A complex conjugate pole pair $H(s) = 2/(s^2 + 2s + 2)$, characterized by $\Omega_n = \sqrt{2}$ radians/sec. and a damping ratio of $\sqrt{2}/2$, and
4. A single real zero term $H_3(s) = (10s + 1)$, with a break frequency of $\Omega = 0.1$ radians/sec.

The component terms are plotted and are added together to determine the total response for a frequency range of 0.01 to 100 radians/sec. in the magnitude and phase plots of Fig. 17.

7 Frequency Response and the Pole-Zero Plot

The frequency response may be written in terms of the system poles and zeros by substituting directly into the factored form of the transfer function:

$$H(j\Omega) = K\frac{(j\Omega - z_1)(j\Omega - z_2)\ldots(j\Omega - z_{m-1})(j\Omega - z_m)}{(j\Omega - p_1)(j\Omega - p_2)\ldots(j\Omega - p_{n-1})(j\Omega - p_n)}. \qquad (76)$$

Because the frequency response is the transfer function evaluated on the imaginary axis of the s-plane, that is when $s = j\Omega$, the graphical method for evaluating the transfer function may be applied directly to the frequency response. Each of the vectors from the n system poles to a test point $s = j\Omega$ has a magnitude and an angle:

$$|j\Omega - p_i| = \sqrt{\sigma_i^2 + (\Omega - \Omega_i)^2}, \qquad (77)$$

$$\angle(s - p_i) = \tan^{-1}\left(\frac{\Omega - \Omega_i}{-\sigma_i}\right), \qquad (78)$$

as shown in Fig. 18a, with similar expressions for the vectors from the m zeros. The magnitude and

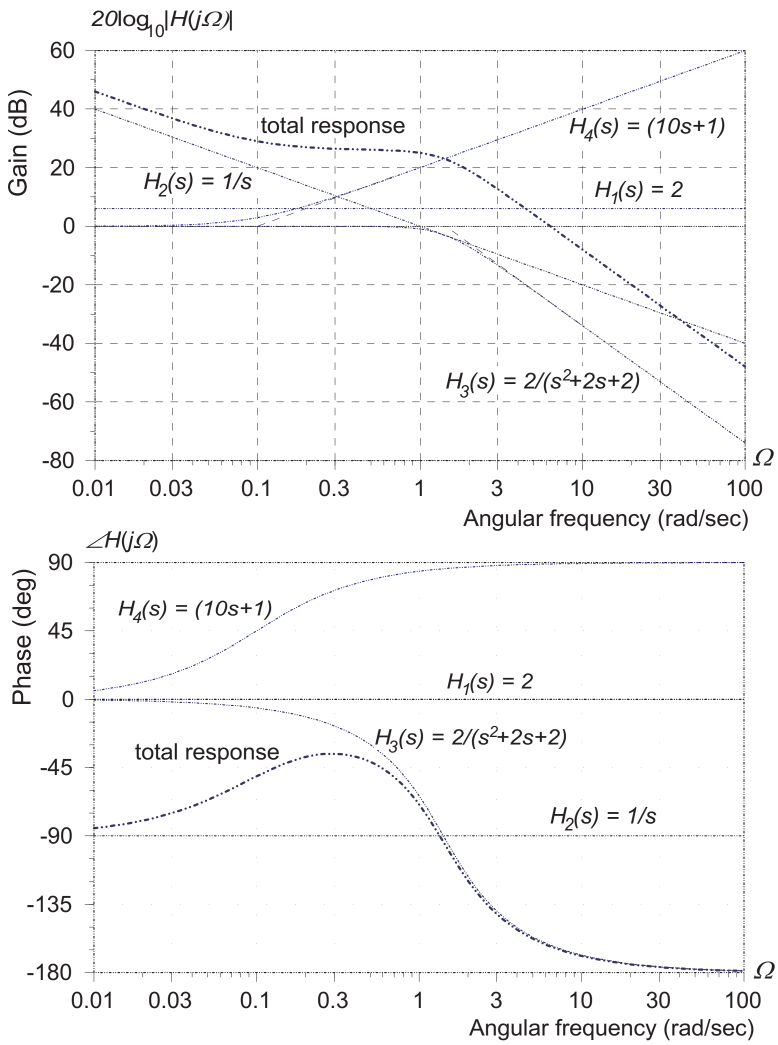

Figure 17: Bode plots for the third-order system of Example 5.

phase angle of the complete frequency response may then be written in terms of the magnitudes and angles of these component vectors

$$|H(j\Omega)| = K\frac{\prod_{i=1}^{m}|(j\Omega - z_i)|}{\prod_{i=1}^{n}|(j\Omega - p_i)|} \quad (79)$$

$$\angle H(j\Omega) = \sum_{i=1}^{m}\angle(j\Omega - z_i) - \sum_{i=1}^{n}\angle(j\Omega - p_i). \quad (80)$$

If the vector from the pole p_i to the point $s = j\Omega$ has length q_i and an angle θ_i from the horizontal, and the vector from the zero z_i to the point $j\Omega$ has a length r_i and an angle ϕ_i, as shown in Fig. 18b, the value of the frequency response at the point $j\Omega$ is

$$|H(j\Omega)| = K\frac{r_1 \ldots r_m}{q_1 \ldots q_n} \quad (81)$$

$$\angle H(j\Omega) = (\phi_1 + \ldots + \phi_m) - (\theta_1 + \ldots + \theta_n) \quad (82)$$

The graphical method can be very useful for deriving a qualitative picture of a system frequency response. For example, consider the sinusoidal response of a first-order system with a pole on the real axis at $s = -1/\tau$ as shown in Fig. 19a, and its Bode plots in Fig. 19b. Even though the gain constant K cannot be determined from the pole-zero plot, the following observations may be made directly by noting the behavior of the magnitude and angle of the vector from the pole to the imaginary axis as the input frequency is varied:

1. At low frequencies the gain approaches a finite value, and the phase angle has a small but finite lag.
2. As the input frequency is increased the gain decreases (because the length of the vector increases), and the phase lag also increases (the angle of the vector becomes larger).
3. At very high input frequencies the gain approaches zero, and the phase angle approaches $\pi/2$.

As a second example consider a second-order system, with the damping ratio chosen so that the pair of complex conjugate poles are located close to the imaginary axis as shown in Fig. 20a. In this case there are a pair of vectors connecting the two poles to the imaginary axis, and the following conclusions may be drawn by noting how the lengths and angles of the vectors change as the test frequency moves up the imaginary axis:

1. At low frequencies there is a finite (but undetermined) gain and a small but finite phase lag associated with the system.
2. As the input frequency is increased and the test point on the imaginary axis approaches the pole, one of the vectors (associated with the pole in the second quadrant) decreases in length and at some point reaches a minimum. There is an increase in the value of the magnitude function over a range of frequencies close to the pole.
3. At very high frequencies, the lengths of both vectors tend to infinity, and the magnitude of the frequency response tends to zero, while the phase approaches an angle of π radians because the angle of each vector approaches $\pi/2$.

The following generalizations may be made about the sinusoidal frequency response of a linear system, based upon the geometric interpretation of the pole-zero plot:

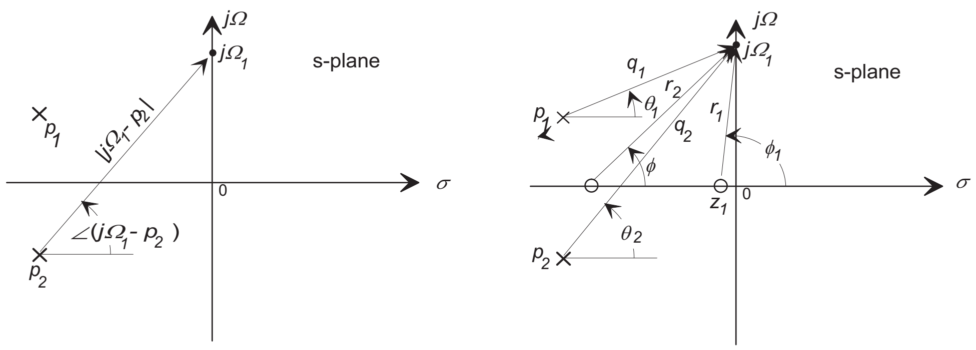

Figure 18: Definition of the vector quantities used in defining the frequency response function from the pole-zero plot. In (a) the vector from a pole (or zero) is defined, in (b) the vectors from all poles and zeros in a typical system are shown.

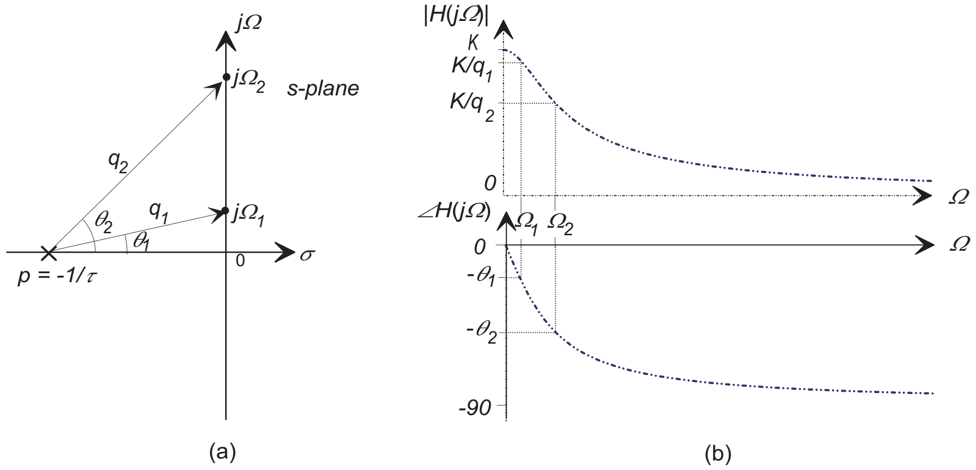

Figure 19: The pole-zero plot of a first-order system and its frequency response functions.

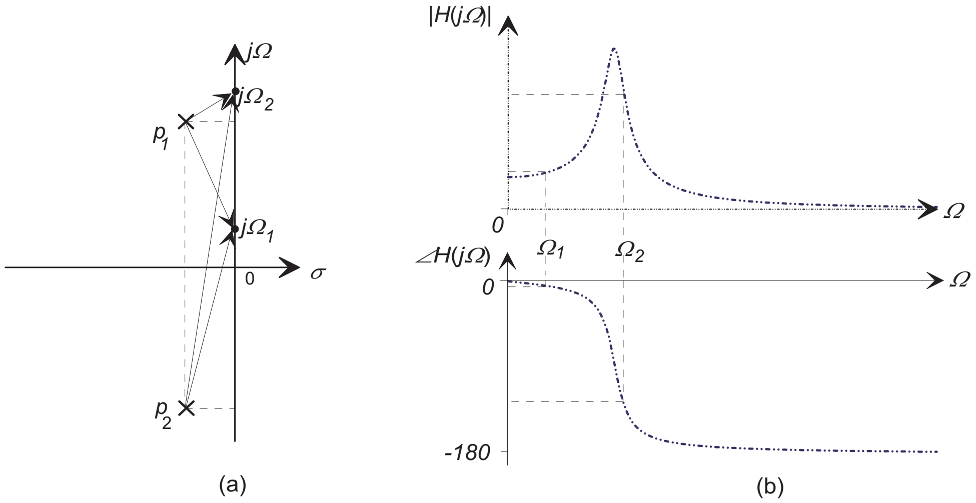

Figure 20: The pole-zero plot for a second-order system and its its frequency response functions.

1. If a system has an excess of poles over the number of zeros ($n > m$) the magnitude of the frequency response tends to zero as the frequency becomes large. Similarly, if a system has an excess of zeros ($n < m$) the gain increases without bound as the frequency of the input increases. (This cannot happen in physical energetic systems because it implies an infinite power gain through the system.) If $n = m$ the system gain becomes constant at high frequencies.

2. If a system has a pair of complex conjugate poles close to the imaginary axis, the magnitude of the frequency response has a "peak", or resonance, at frequencies in the proximity of the pole. If the pole pair lies directly upon the imaginary axis, the system exhibits an infinite gain at that frequency.

3. If a system has a pair of complex conjugate zeros close to the imaginary axis, the frequency response has a "dip" or "notch" in its magnitude function at frequencies in the vicinity of the

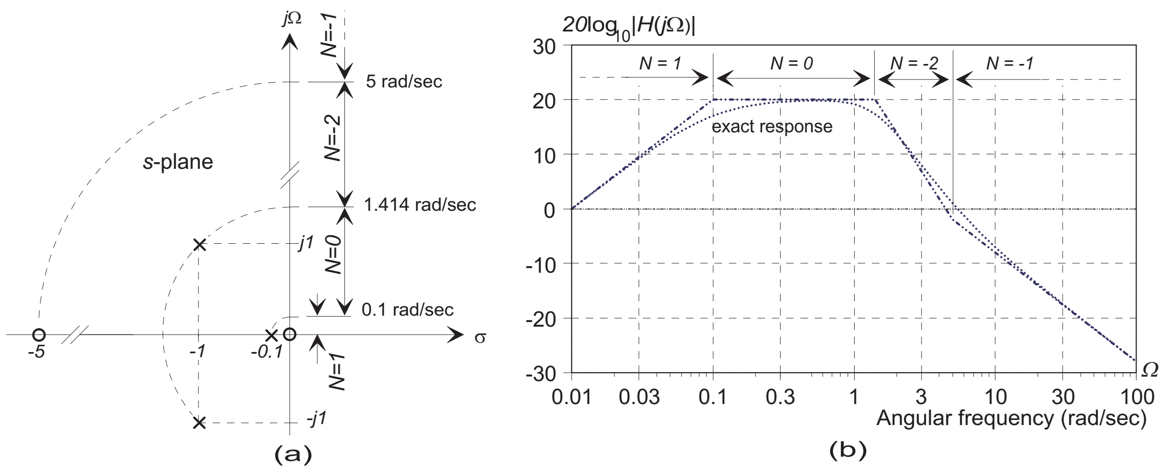

Figure 21: Construction of the magnitude Bode plot from the pole-zero diagram: (a) shows a typical third-order system, and the definition of the break frequencies, (b) shows the Bode plot based on changes in slope at the break frequencies

zero. Should the pair of zeros lie directly upon the imaginary axis, the response is identically zero at the frequency of the zero, and the system does not respond at all to sinusoidal excitation at that frequency.

4. A pole at the origin of the s-plane (corresponding to a pure integration term in the transfer function) implies an infinite gain at zero frequency.

5. Similarly a zero at the origin of the s-plane (corresponding to a pure differentiation) implies a zero gain for the system at zero frequency.

7.1 A Simple Method for constructing the Magnitude Bode Plot directly from the Pole-Zero Plot

The pole-zero plot of a system contains sufficient information to define the frequency response except for an arbitrary gain constant. It is often sufficient to know the shape of the magnitude Bode plot without knowing the absolute gain. The method described here allows the magnitude plot to be sketched by inspection, without drawing the individual component curves. The method is based on the fact that the overall magnitude curve undergoes a *change* in slope at each break frequency.

The first step is to identify the break frequencies, either by factoring the transfer function or directly from the pole-zero plot. Consider a typical pole-zero plot of a linear system as shown in Fig. 21a. The break frequencies for the four first and second-order blocks are all at a frequency equal to the radial distance of the poles or zeros from the origin of the s-plane, that is $\omega_b = \sqrt{\sigma^2 + \omega^2}$. Therefore all break frequencies may be found by taking a compass and drawing an arc from each pole or zero to the positive imaginary axis. These break frequencies may be transferred directly to the logarithmic frequency axis of the Bode plot.

Because all low frequency asymptotes are horizontal lines with a gain of 0dB, a pole or zero does not contribute to the magnitude Bode plot below its break frequency. Each pole or zero contributes a change in the *slope* of the asymptotic plot of ±20 dB/decade above its break frequency. A complex conjugate pole or zero pair defines *two* coincident breaks of ±20 dB/decade (one from each member of the pair), giving a total change in the slope of ± 40 dB/decade. Therefore, at any frequency Ω, the slope of the asymptotic magnitude function depends only on the number of break points at frequencies less than Ω, or to the left on the Bode plot. If there are Z breakpoints due to zeros to the left, and P breakpoints due to poles, the slope of the curve at that frequency is $20 \times (Z - P)$ dB/decade.

Any poles or zeros at the origin cannot be plotted on the Bode plot, because they are effectively to the left of all finite break frequencies. However, they define the initial slope. If an arbitrary starting frequency and an assumed gain (for example 0dB) at that frequency are chosen, the shape of the magnitude plot may be easily constructed by noting the initial slope, and constructing the curve from straight line segments that change in slope by units of ± 20 dB/decade at the breakpoints. The arbitrary choice of the reference gain results in a vertical displacement of the curve.

Figure 21b shows the straight line magnitude plot for the system shown in Fig. 21a constructed using this method. A frequency range of 0.01 to 100 radians/sec was arbitrarily selected, and a gain of 0dB at 0.01 radians/sec was assigned as the reference level. The break frequencies at 0, 0.1, 1.414, and 5 radians/sec were transferred to the frequency axis from the pole-zero plot. The value of N at any frequency is $Z - P$, where Z is the number of zeros to the left, and P is the number of poles to the left. The curve was simply drawn by assigning the value of the slope in each of the frequency intervals and drawing connected lines.

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