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Lecture 2

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Contents

2 Differential geometry of curves **2**
- 2.1 Definition of curves 2
  - 2.1.1 Plane curves 2
  - 2.1.2 Space curves 4
- 2.2 Arc length 7
- 2.3 Tangent vector 8
- 2.4 Normal vector and curvature 10
- 2.5 Binormal vector and torsion 12
- 2.6 Serret-Frenet Formulae 13

Bibliography **16**

Reading in the Textbook

- Chapter 1, pp.1 - pp.3
- Chapter 2, pp.36 - pp.48

Lecture 2

Differential geometry of curves

2.1 Definition of curves

2.1.1 Plane curves

- *Implicit curves* $f(x, y) = 0$
  Example: $x^2 + y^2 = a^2$

  - It is difficult to trace implicit curves.
  - It is easy to check if a point lies on the curve.
  - Multi-valued and closed curves can be represented.
  - It is easy to evaluate tangent line to the curve when the curve has a vertical or near vertical tangent.
  - Axis dependent. (Difficult to transform to another coordinate system).

  Example: $x^3 + y^3 = 3xy$: Folium of Descartes (see Figure 2.1a)

  $$\begin{aligned} \text{Let } f(x, y) &= x^3 + y^3 - 3xy = 0, \\ f(0, 0) &= 0 \Rightarrow (x, y) = (0, 0) \quad \text{lies on the curve} \end{aligned}$$

  Example: If we translate by (1,2) and rotate the axes by $\theta = atan(\frac{3}{4})$, the hyperbola $\frac{x^2}{4} - \frac{y^2}{2} = 1$, shown in Figure 2.1(b), will become $2x^2 - 72xy + 23y^2 + 140x - 20y + 50 = 0$.

- *Explicit curves* $y = f(x)$
  One of the variables is expressed in terms of the other.
  Example: $y = x^2$

  - It is easy to trace explicit curves.
  - It is easy to check if a point lies on the curve.
  - Multi-valued and closed curves can not be easily represented.
  - It is difficult to evaluate tangent line to the curve when the curve has a vertical or near vertical tangent.

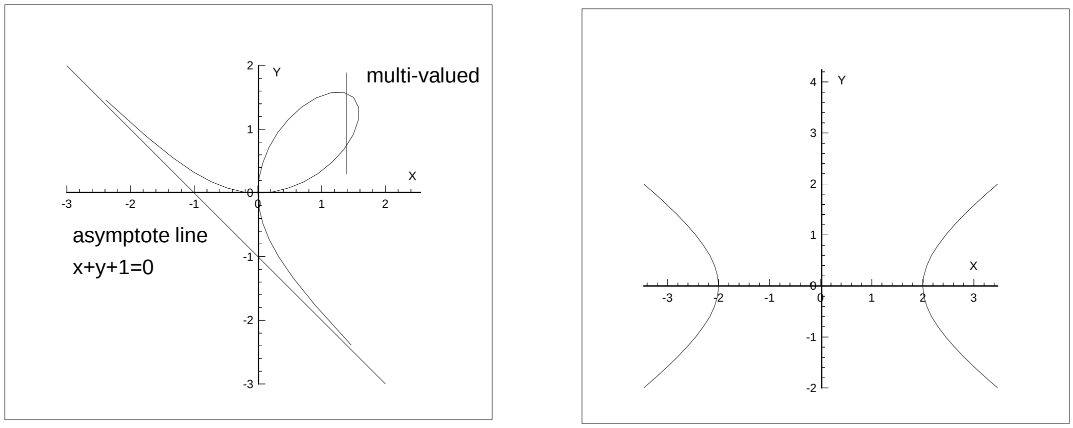

Figure 2.1: (a) Descartes; (b) Hyperbola.

– Axis dependent. (Difficult to transform to another coordinate system).

Example: If the circle is represented by an explicit equation, it must be divided into two segments, with $y = +\sqrt{r^2 - x^2}$ for the upper half and $y = -\sqrt{r^2 - x^2}$ for the lower half, see Figure 2.2. This kind of segmentation creates cases which are inconvenient in computer programming and graphics.

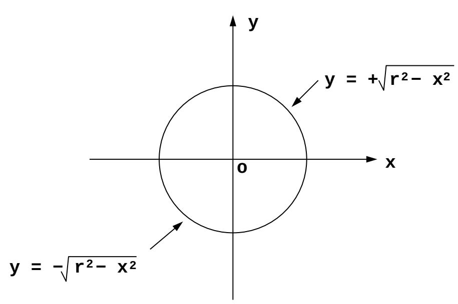

Figure 2.2: Description of a circle with an explicit equation.

Note: The derivative of $y = \sqrt{x}$ at the origin $x = 0$ is infinite, see Figure 2.3.

- *Parametric curves* $x = x(t),\ y = y(t),\ t_1 \leq t \leq t_2$

2D coordinates (x, y) can be expressed as functions of a parameter t.

Example: $x = a\cos(t),\ \ y = a\sin(t),\ \ 0 \leq t < 2\pi$

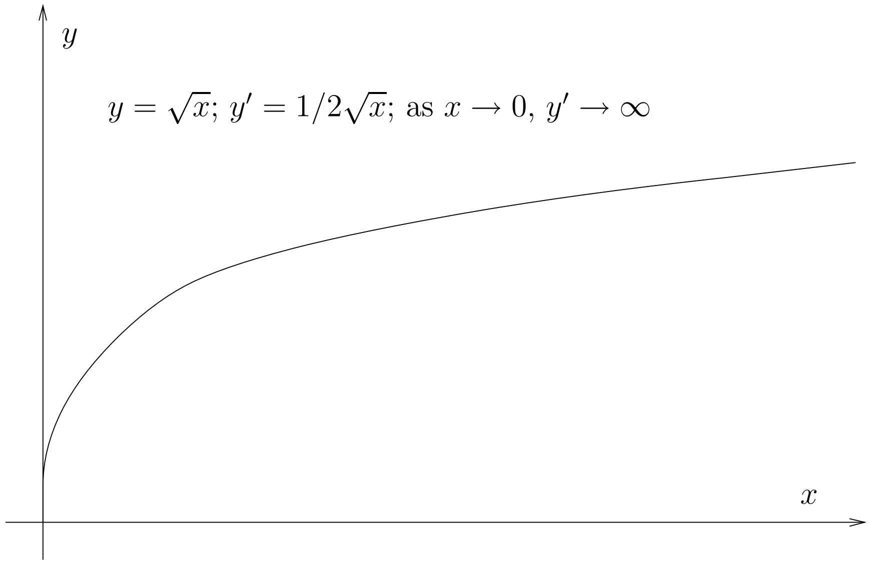

Figure 2.3: Vertical slopes for explicit curves involve non-polynomial functions.

- It is easy to trace parametric curves.
- It is relatively difficult to check if a point lies on the curve.
- Closed and multi-valued curves are easy to represent.
- It is easy to evaluate tangent line to the curve when the curve has a vertical or near vertical tangent.
- Axis independent. (Easy to transform to another coordinate system)

Example: Folium of Descartes, see Figure 2.1, can be expressed as:
$\mathbf{r}(t) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3}\right) \quad -\infty < t < \infty \quad \Rightarrow$ easy to trace
$x(t) = x_0 \Rightarrow$ solve for $t \Rightarrow$ plug t into $y(t) = y_0 \quad \Rightarrow$ need to solve a nonlinear equation to check if a point lies on the curve.

Explicit curve $y = \sqrt{x}$ can be expressed as $x = t^2$, $y = t$ ($t \geq 0$).

$$\mathbf{r} = (t^2, t), \qquad \dot{\mathbf{r}} = (2t, 1)$$
$$\text{unit tangent vector} \quad \mathbf{t} = \frac{(2t, 1)}{\sqrt{4t^2 + 1}}$$
$$\text{at} \;\; t = 0, \mathbf{t} = (0, 1)$$

Therefore there is no problem representing a vertical tangent computationally.

2.1.2 Space curves

- *Implicit curves*
  In 3D, a single equation generally represents a surface. For example $x^2 + y^2 + z^2 = a^2$ is a sphere.

Thus, the curve appears as the intersection of two surfaces.

$$F(x, y, z) = 0 \cap G(x, y, z) = 0$$

Example: Intersection of the two quadric surfaces $z = xy$ and $y^2 = zx$ gives cubic parabola. (These two surfaces intersect not only along the cubic parabola but also along the x-axis.)

- *Explicit curves*
  If the implicit equations can be solved for two of the variables in terms of the third, say for y and z in terms of x, we get

  $$y = y(x), \quad z = z(x)$$

  Each of the equations separately represents a cylinder projecting the curve onto one of the coordinate planes. Therefore intersection of the two cylinders represents the curve.

  Example: Intersection of the two cylinders $y = x^2$, $z = x^3$ gives a cubic parabola.

- *Parametric curves* $x = x(t),\ y = y(t),\ z = z(t),\ t_1 \le t \le t_2$
  The 3D coordinates (x, y, z) of the point can be expressed on functions of parameter t. Here functions $x(t)$, $y(t)$, $z(t)$ have continuous derivatives of the rth order, and the parameter t is restricted to some interval called the parameter space (i.e., $t_1 \le t \le t_2$). In this case the curve is said to be of class r, denoted as C^r.

  In vector notation:

  $$\begin{aligned} &\mathbf{r} = \mathbf{r}(t) \\ &where \quad \mathbf{r} = (x, y, z), \quad \mathbf{r}(t) = (x(t), y(t), z(t)) \end{aligned}$$

  Example: Cubic parabola

  $$x = t, \qquad y = t^2, \qquad z = t^3$$

  Example: Circular helix, see Fig. 2.4.

  $$x = a\cos(t), \qquad y = a\sin(t), \qquad z = bt, \qquad 0 \le t \le \pi$$

  Using $v = \tan\frac{t}{2}$

  $$\begin{aligned} v = tan\frac{t}{2} = \sqrt{\frac{1-\cos t}{1+\cos t}} &\Rightarrow v^2 = \frac{1-\cos t}{1+\cos t} \\ \Rightarrow \cos t = \frac{1-v^2}{1+v^2} &\Rightarrow \sin t = \frac{2v}{1+v^2} \end{aligned}$$

  Therefore the following parametrization will give the same circular helix.

  $$\mathbf{r} = \left(a\frac{1-v^2}{1+v^2}, \quad \frac{2av}{1+v^2}, \quad 2btan^{-1}v \right), \quad 0 \le v < \infty$$

```
>>
>> a= 2;
>> b = 3;
>> u = [0 : 6 * pi / 100 : 6 * pi];
>> plot3(a * cos(u), a * sin(u), b * u)
>> xlabel('X');
>> ylabel('Y');
>> zlabel('Z');
>> print('circHelix.ps')
```

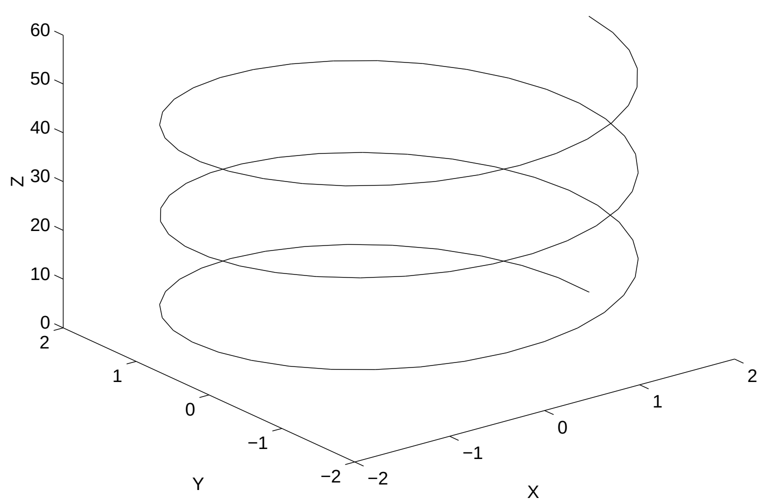

Figure 2.4: Circular helix plotted using MATLAB.

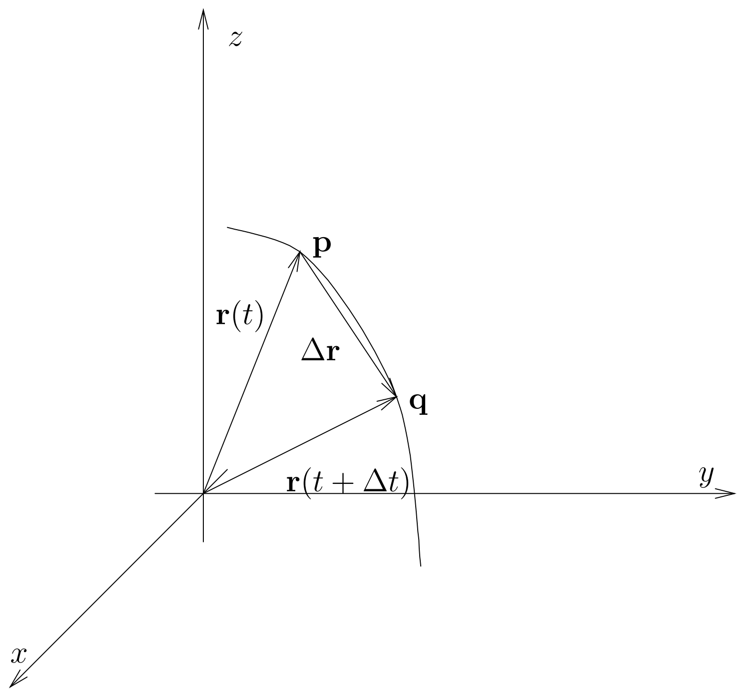

Figure 2.5: A segment $\Delta\mathbf{r}$ connecting two point $\mathbf{p}$ and $\mathbf{q}$ on a parametric curve $\mathbf{r}(t)$.

2.2 Arc length

From Figure 2.5, we will derive an expression for the differential arc length ds of a parametric curve. First, let us express the vector $\Delta\mathbf{r}$ connecting two points $\mathbf{p}$ and $\mathbf{q}$ on an arc at parametric locations t and $t + \Delta t$, respectively, as

$$\Delta\mathbf{r} = \mathbf{p} - \mathbf{q} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t).$$

As $\mathbf{p}$ and $\mathbf{q}$ become infinitesimally close, the length of the segment connecting the two points approaches the arc length between the two points along the curve, $\mathbf{r}(t)$ and $\mathbf{r}(t+\Delta t)$. Or using Taylor's expansion on the norm (length) of the segment $\Delta\mathbf{r}$ and letting $\Delta t \to 0$, we can express the differential arc length as

$$\Delta s \simeq |\Delta\mathbf{r}| = |\mathbf{r}(t + \Delta t) - \mathbf{r}(t)| = |\frac{d\mathbf{r}}{dt}\Delta t + O(\Delta t^2)| \simeq |\frac{d\mathbf{r}}{dt}|\Delta t.$$

Thus as $\Delta t \to 0$

$$ds = |\frac{d\mathbf{r}}{dt}|dt = |\dot{\mathbf{r}}|dt.$$

Definitions

$$\frac{d}{dt} \equiv \dot{}$$
$$\frac{d}{ds} \equiv \, '$$

Hence the rate of change $\frac{ds}{dt}$ of the arc length s with respect to the parameter t is

$$\frac{ds}{dt} = \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)} \quad (2.1)$$

$\frac{ds}{dt}$ is called the *parametric speed*. It is, by definition, non-negative (s being measured always in the sense of increasing t).

If the parametric speed does not vary significantly, parameter values $t_0, t_1, \cdots, t_N$ corresponding to a uniform increment $\Delta t = t_k - t_{k-1}$, will be evenly distributed along the curve, as illustrated in Figure 2.6.

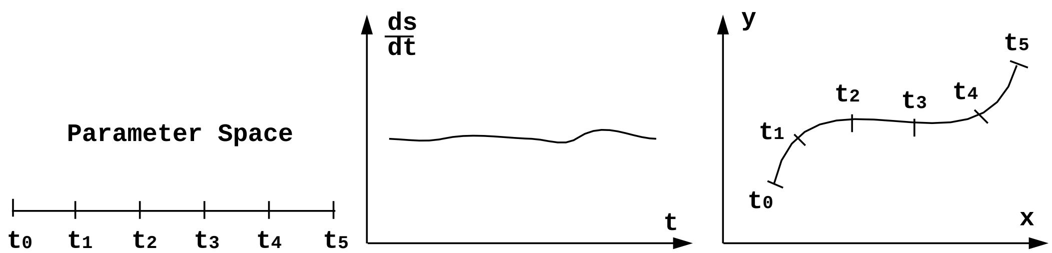

Figure 2.6: When parametric speed does not vary, parameter values are uniformly spaced along a parametric curve.

The arc length of a segment of the curve between points $\mathbf{r}(t_o)$ and $\mathbf{r}(t)$ can be obtained as follows:

$$s(t) = \int_{t_o}^{t} \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)}dt = \int_{t_o}^{t} \sqrt{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}dt \tag{2.2}$$

Derivatives of arc length s w.r.t. parameter t and vice versa :

$$\dot{s} = \frac{ds}{dt} = |\dot{\mathbf{r}}| = \sqrt{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}} \tag{2.3}$$

$$\ddot{s} = \frac{d\dot{s}}{dt} = \frac{\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}}}{\sqrt{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}} \tag{2.4}$$

$$\dddot{s} = \frac{d\ddot{s}}{dt} = \frac{(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})(\dot{\mathbf{r}} \cdot \dddot{\mathbf{r}} + \ddot{\mathbf{r}} \cdot \ddot{\mathbf{r}}) - (\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}})^2}{(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})^{\frac{3}{2}}} \tag{2.5}$$

$$t' = \frac{dt}{ds} = \frac{1}{|\dot{\mathbf{r}}|} = \frac{1}{\sqrt{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}} \tag{2.6}$$

$$t'' = \frac{dt'}{ds} = -\frac{\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}}}{(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})^2} \tag{2.7}$$

$$t''' = \frac{dt''}{ds} = -\frac{(\ddot{\mathbf{r}} \cdot \ddot{\mathbf{r}} + \dot{\mathbf{r}} \cdot \dddot{\mathbf{r}})(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}) - 4(\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}})^2}{(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})^{\frac{7}{2}}} \tag{2.8}$$

2.3 Tangent vector

The vector $\mathbf{r}(t+\Delta t) - \mathbf{r}(t)$ indicates the direction from $\mathbf{r}(t)$ to $\mathbf{r}(t+\Delta t)$. If we divide the vector by Δt and take the limit as $\Delta t \to 0$, then the vector will converge to the finite magnitude vector $\dot{\mathbf{r}}(t)$.

$\dot{\mathbf{r}}(t)$ is called the *tangent vector*.

Magnitude of the tangent vector

$$|\dot{\mathbf{r}}| = \frac{ds}{dt} \tag{2.9}$$

Unit tangent vector

$$\mathbf{t} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} = \frac{\frac{d\mathbf{r}}{dt}}{\frac{ds}{dt}} = \frac{d\mathbf{r}}{ds} \equiv \mathbf{r}' \tag{2.10}$$

Definition : A parametric curve is said to be *regular* if $|\dot{\mathbf{r}}(t)| \neq 0$ for all $t \in I$. The points where $|\dot{\mathbf{r}}(t)| = 0$ are called irregular (singular) points.

Note that at irregular points the parametric speed is zero.

Example: semi-cubical parabola $\mathbf{r}(t) = (t^2, t^3)$, see Figure 2.7

$$\begin{aligned}\dot{\mathbf{r}}(t) &= (2t, 3t^2) \\ |\dot{\mathbf{r}}(t)| &= \sqrt{4t^2 + 9t^4} = \sqrt{t^2(4 + 9t^2)}\end{aligned}$$

$$\text{when } \ t = 0, |\dot{\mathbf{r}}(t)| = 0$$

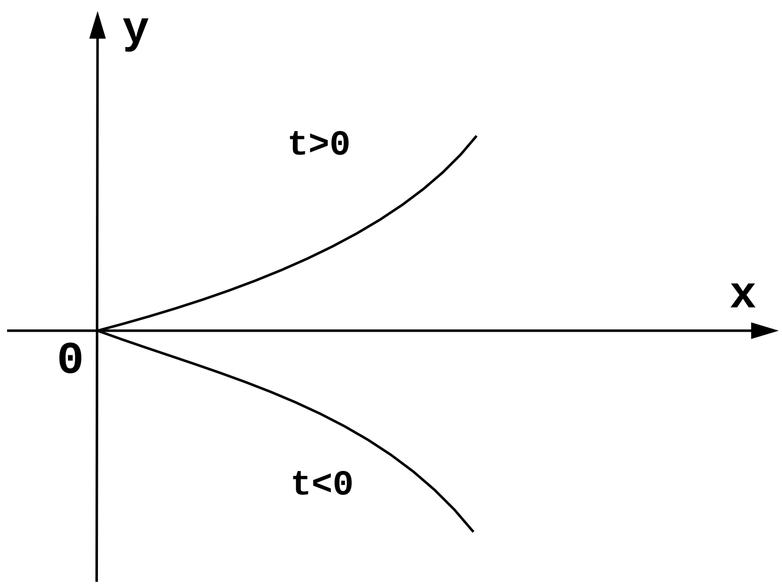

Figure 2.7: A singular point occurs on a semi-cubical parabola in the form of a cusp.

Here are some useful formulae for computing the unit tangent vector:

- 3D Parametric curve $\mathbf{r}(t)$

$$\mathbf{t} \;=\; \mathbf{r}' = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} = \frac{(\dot{x}, \dot{y}, \dot{z})}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}$$

- 2D Implicit curve $f(x, y) = 0$

$$\mathbf{t} = \frac{(f_y, -f_x)}{\sqrt{f_x^2 + f_y^2}}$$

- 2D Explicit curve $y = f(x)$

$$\mathbf{t} = \frac{(1, \dot{f})}{\sqrt{1+\dot{f}^2}}$$

Example: For a circular helix $\mathbf{r}(t) = (a\cos t, a\sin t, bt)$

- Parametric speed

$$\begin{aligned} \frac{ds}{dt} &= |\dot{\mathbf{r}}(t)| = \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)} \\ \dot{\mathbf{r}}(t) &= (-a\sin t, a\cos t, b) \\ |\dot{\mathbf{r}}(t)| &= \sqrt{a^2+b^2} = c = \text{const} \Rightarrow \begin{cases} \text{The curve is regular and has} \\ \text{good parametrization} \end{cases} \end{aligned}$$

- Unit tangent vector

$$\mathbf{t} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} = (-\frac{a}{c}\sin t, \frac{a}{c}\cos t, \frac{b}{c}) \tag{2.11}$$

- Arc length

$$s(t) = \int_0^t |\dot{\mathbf{r}}| dt = \int_0^t \sqrt{a^2+b^2} dt = ct \tag{2.12}$$

- Arc length parametrization

$$t = \frac{s}{c} \tag{2.13}$$

$$\mathbf{r}(s) = (a\cos\frac{s}{c}, a\sin\frac{s}{c}, \frac{bs}{c}) \tag{2.14}$$

$$\text{check} \tag{2.15}$$

$$\frac{d\mathbf{r}}{ds} = (-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}) = \mathbf{t} \tag{2.16}$$

2.4 Normal vector and curvature

Let us consider the second derivative $\mathbf{r}''(s)$, see Figure 2.8.

$$\mathbf{r}''(s) = \lim_{\Delta s \to 0} \frac{\mathbf{r}'(s+\Delta s) - \mathbf{r}'(s)}{\Delta s} \tag{2.17}$$

As $\Delta s \to 0$ $\mathbf{r}'(s+\Delta s) - \mathbf{r}'(s)$ becomes perpendicular to the tangent vector i.e. normal direction.

Also $|\mathbf{r}'(s+\Delta s) - \mathbf{r}'(s)| = \Delta\theta \cdot 1 = \Delta\theta$ as $\Delta s \to 0$.

Thus

$$|\mathbf{r}''(s)| = \lim_{\Delta s \to 0} \frac{\Delta\theta}{\Delta s} = \lim_{\Delta s \to 0} \frac{\frac{\Delta\theta}{\rho}}{\Delta\theta} = \frac{1}{\rho} \equiv \kappa \tag{2.18}$$

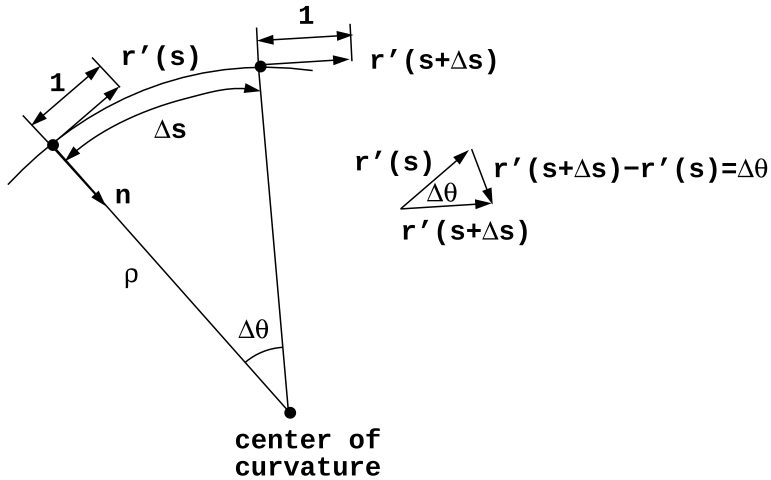

Figure 2.8: Derivation of the normal vector of a curve.

κ is called the curvature. It follows that

$$\kappa^2 = \mathbf{r}'' \cdot \mathbf{r}''. \tag{2.19}$$

Consequently

$$\mathbf{r}''(s) = \mathbf{t}' = \kappa \mathbf{n} \tag{2.20}$$

Thus using equations (2.6) and (2.7), we obtain

$$\kappa \mathbf{n} = \frac{d^2\mathbf{r}}{ds^2} = \frac{d\mathbf{t}}{ds} = \frac{d}{ds}(\dot{\mathbf{r}}t') = \ddot{\mathbf{r}}(t')^2 + \dot{\mathbf{r}}t'' = \frac{(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})\ddot{\mathbf{r}} - (\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}})\dot{\mathbf{r}}}{(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})^2} \tag{2.21}$$

$$\kappa^2 = (\kappa\mathbf{n})\cdot(\kappa\mathbf{n}) = \left[\frac{(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})\ddot{\mathbf{r}} - (\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}})\dot{\mathbf{r}}}{(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})^2}\right]\cdot\left[\frac{(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})\ddot{\mathbf{r}} - (\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}})\dot{\mathbf{r}}}{(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})^2}\right] = \frac{(\dot{\mathbf{r}}\times\ddot{\mathbf{r}})\cdot(\dot{\mathbf{r}}\times\ddot{\mathbf{r}})}{(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})^3} \tag{2.22}$$

where the identity $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b}) = (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})^2$ is used.

Here are some useful formulae for computing the normal vector and curvature:

- 2D parametric curve $\mathbf{r}(t)$, see Figure 2.9

$$\mathbf{n} = \mathbf{e}_z \times \mathbf{t} = \frac{(-\dot{y}, \dot{x})}{\sqrt{\dot{x}^2 + \dot{y}^2}}, \quad \mathbf{e}_z = (0, 0, 1) \tag{2.23}$$

$$\kappa = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}} \tag{2.24}$$

- 2D implicit curve $f(x, y) = 0$

$$\mathbf{n} = \mathbf{e}_z \times \mathbf{t} = \frac{(f_x, f_y)}{\sqrt{f_x^2 + f_y^2}} = \frac{\nabla f}{|\nabla f|} \tag{2.25}$$

$$\kappa = -\frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_x^2 f_{yy}}{(f_x^2 + f_y^2)^{\frac{3}{2}}} \tag{2.26}$$

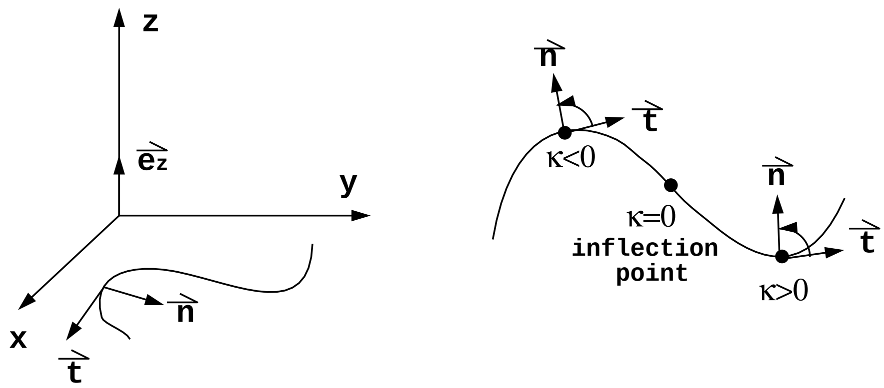

Figure 2.9: Normal and tangent vectors along a 2D curve.

- 2D Explicit curve $y = f(x)$

$$\mathbf{n} = \mathbf{e}_z \times \mathbf{t} = \frac{(-\dot{y}, 1)}{\sqrt{1+\dot{y}^2}} \tag{2.27}$$

$$\kappa = \frac{\ddot{y}}{(1+\dot{y}^2)^{\frac{3}{2}}} \tag{2.28}$$

2.5 Binormal vector and torsion

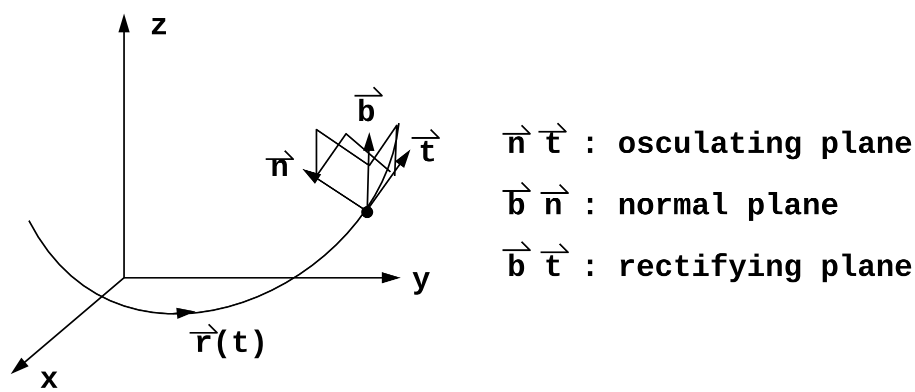

Figure 2.10: The tangent, normal, and binormal vectors define an orthogonal coordinate system along a space curve.

Let us define a unit binormal vector, see Figure 2.10

$$\mathbf{b} = \mathbf{t} \times \mathbf{n} \tag{2.29}$$

We have

$$\mathbf{t}\cdot\mathbf{n} = 0 \qquad \mathbf{n}\cdot\mathbf{b} = 0 \qquad \mathbf{b}\cdot\mathbf{t} = 0$$
$$\mathbf{b} = \mathbf{t}\times\mathbf{n} \qquad \mathbf{t} = \mathbf{n}\times\mathbf{b} \qquad \mathbf{n} = \mathbf{b}\times\mathbf{t}$$

The osculating plane can be defined as the plane passing through three consecutive points on the curve. The rate of change of the osculating plane is expressed by the vector

$$\mathbf{b}' = \frac{d}{ds}(\mathbf{t} \times \mathbf{n}) = \frac{d\mathbf{t}}{ds} \times \mathbf{n} + \mathbf{t} \times \frac{d\mathbf{n}}{ds} = \mathbf{t} \times \mathbf{n}' \tag{2.30}$$

where we used the fact that $\frac{d\mathbf{t}}{ds} = \mathbf{r}'' = \kappa\mathbf{n}$.

From $\mathbf{n} \cdot \mathbf{n} = 1 \rightarrow$ differentiate w.r.t. $s \rightarrow 2\mathbf{n}' \cdot \mathbf{n} = 0 \rightarrow \mathbf{n}' \perp \mathbf{n}$

Thus $\mathbf{n}'$ is parallel to the rectifying plane $(\mathbf{b}, \mathbf{t})$, and $\mathbf{n}'$ can be expressed as a linear combination of $\mathbf{b}$ and $\mathbf{t}$.

$$\mathbf{n}' = \mu\mathbf{t} + \tau\mathbf{b} \tag{2.31}$$

Substitute (2.31) into (2.30)

$$\mathbf{b}' = \mathbf{t} \times (\mu\mathbf{t} + \tau\mathbf{b}) = \tau\mathbf{t} \times \mathbf{b} = -\tau\mathbf{b} \times \mathbf{t} = -\tau\mathbf{n} \tag{2.32}$$

τ is called the *torsion*.

Consequently

$$\tau = -\mathbf{n} \cdot \mathbf{b}' = -\mathbf{n} \cdot (\mathbf{t} \times \mathbf{n})' = \frac{(\mathbf{r}'\mathbf{r}''\mathbf{r}''')}{\mathbf{r}'' \cdot \mathbf{r}''} = \frac{(\dot{\mathbf{r}}\ddot{\mathbf{r}}\,\dddot{\mathbf{r}})}{(\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) \cdot (\dot{\mathbf{r}} \times \ddot{\mathbf{r}})} \tag{2.33}$$

Triple scalar product

$$(\mathbf{abc}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \tag{2.34}$$

also

$$(\mathbf{abc}) = (\mathbf{bca}) = (\mathbf{cab}) \quad \text{cyclic permutation} \tag{2.35}$$

Geometrically, $(\mathbf{abc})$ equals to the volume of a parallelepiped having the edge vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, as in Figure 2.11.

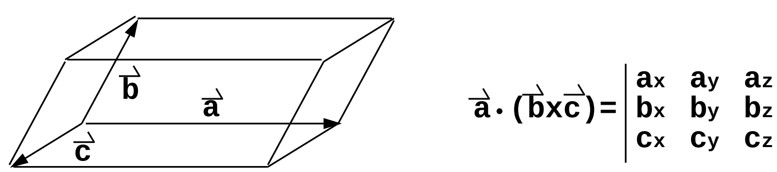

Figure 2.11: The computation of the volume of a parallelepiped

2.6 Serret-Frenet Formulae

From equations (2.20) and (2.32), we found that

$$\mathbf{t}' = \kappa\mathbf{n} \tag{2.36}$$

$$\mathbf{b}' = -\tau\mathbf{n} \tag{2.37}$$

How about $\mathbf{n}'$?

$$\mathbf{n}' = (\mathbf{b} \times \mathbf{t})' = \mathbf{b}' \times \mathbf{t} + \mathbf{b} \times \mathbf{t}' = -\tau \mathbf{n} \times \mathbf{t} + \mathbf{b} \times (\kappa \mathbf{n}) = -\kappa \mathbf{t} + \tau \mathbf{b} \tag{2.38}$$

In matrix form we can express the differential equations as

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} \tag{2.39}$$

Thus, the curve is completely determined by its curvature and torsion as a function of parameter s. The equations $\kappa = \kappa(s)$, $\tau = \tau(s)$ are called *intrinsic equations.* The formulae 2.39 are known as the Serret-Frenet Formulae and describe the motion of moving a trihedron ($\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$) along the curve.

Example: Determining the shape of a curve from curvature information and boundary conditions only.

Given:

$$\kappa = \frac{1}{R} = \text{const}$$

We find

$$\frac{d\mathbf{t}}{ds} = \frac{\mathbf{n}}{R} \tag{2.40}$$

$$\frac{d\mathbf{n}}{ds} = -\frac{\mathbf{t}}{R} \tag{2.41}$$

If we diffrentiate Equation 2.40 with respect to s,

$$\frac{d^2\mathbf{t}}{ds^2} = \frac{1}{R}\frac{d\mathbf{n}}{ds}. \tag{2.42}$$

Now, substitute Equation 2.42 into Equation 2.41

$$\frac{d^2\mathbf{t}}{ds^2} + \frac{\mathbf{t}}{R^2} = 0. \tag{2.43}$$

Recognizing that $\mathbf{t} = \frac{d\mathbf{r}}{ds}$, we can change variables from $\mathbf{t}$ to $\mathbf{r}$, transforming Equation 2.43 into

$$\frac{d^3\mathbf{r}}{ds^3} + \frac{1}{R^2}\frac{d\mathbf{r}}{ds} = 0$$

or

$$\frac{d^3}{ds^3}\begin{pmatrix} x(s) \\ y(s) \end{pmatrix} + \frac{1}{R^2}\frac{d}{ds}\begin{pmatrix} x(s) \\ y(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.44}$$

The solution to Equation 2.44 is

$$x(s) = C_1 + C_2 \cos\left(\frac{s}{R}\right) + C_3 \sin\left(\frac{s}{R}\right) \tag{2.45}$$

$$y(s) = C_1' + C_2' \cos\left(\frac{s}{R}\right) + C_3' \sin\left(\frac{s}{R}\right) \tag{2.46}$$

Assume we are given suitable initial conditions or boundary conditions. For this example, we will use:

$$x(0) = R \quad x'(0) = 0 \quad x''(0) = -\frac{1}{R} \tag{2.47}$$

$$y(0) = 0 \quad y'(0) = 1 \quad y''(0) = 0 \tag{2.48}$$

Solving for the constants in the general solution gives

$$C_1 = C_3 = 0 \quad C_2 = R \tag{2.49}$$

$$C_1' = C_2' = 0 \quad C_3' = R \tag{2.50}$$

Thus, we find our solution is given by

$$x(s) = R\cos\left(\frac{s}{R}\right) \tag{2.51}$$

$$y(s) = R\sin\left(\frac{s}{R}\right) \tag{2.52}$$

which is precisely a circle of radius R satisfying the conditions (2.47) and (2.48).

Example: A circular helix $\mathbf{r} = (a\cos\frac{s}{c}, a\sin\frac{s}{c}, \frac{bs}{c})$

$$\begin{aligned}
\mathbf{r}'(s) &= (-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}) \\
\mathbf{r}''(s) &= (-\frac{a}{c^2}\cos\frac{s}{c}, -\frac{a}{c^2}\sin\frac{s}{c}, 0) \\
\mathbf{r}'''(s) &= (\frac{a}{c^3}\sin\frac{s}{c}, -\frac{a}{c^3}\cos\frac{s}{c}, 0) \\
\kappa^2 &= \mathbf{r}''\cdot\mathbf{r}'' = \frac{a^2}{c^4}(\cos^2\frac{s}{c} + \sin^2\frac{s}{c}) = \frac{a^2}{c^4} = \mathit{constant} \\
\tau &= \frac{(\mathbf{r}'\mathbf{r}''\mathbf{r}''')}{\mathbf{r}''\cdot\mathbf{r}''} = \frac{(\mathbf{r}'\mathbf{r}''\mathbf{r}''')}{\kappa^2} \\
&= \frac{c^4}{a^2}\begin{vmatrix} -\frac{a}{c}\sin\frac{s}{c} & \frac{a}{c}\cos\frac{s}{c} & \frac{b}{c} \\ -\frac{a}{c^2}\cos\frac{s}{c} & -\frac{a}{c^2}\sin\frac{s}{c} & 0 \\ \frac{a}{c^3}\sin\frac{s}{c} & -\frac{a}{c^3}\cos\frac{s}{c} & 0 \end{vmatrix} \\
&= \frac{c^4}{a^2}\frac{b}{c}(\frac{a^2}{c^5}(\cos^2\frac{s}{c} + \sin^2\frac{s}{c})) \\
&= \frac{b}{c^2} = \mathit{constant}
\end{aligned}$$

Note: when $b > 0$, it is a right-handed helix;
when $b < 0$, it is a left-handed helix.

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