Choice of State Variables

It must be kept in mind that there is no unique set of state variables (though as we will see, in many cases, some choices are clearly superior to others).

Referring again to the system depicted in figure 4.11 and assuming an ideal (linear) capacitor and inertia, their constitutive equations may be differentiated as follows.

$$F_c = k\, x_c \tag{4.80}$$

$$dF_c/dt = k\, v_c \tag{4.81}$$

$$v_i = p_i/m \tag{4.82}$$

$$dv_i/dt = F_i/m \tag{4.83}$$

This suggests that F_c and v_i would be a reasonable choice for state variables. Equations 4.81 and 4.83 may be combined using the constitutive equations of the one-junction.

$$v_c = v_i \tag{4.84}$$

$$F_i = -F_c \tag{4.85}$$

The following state equations result.

$$\frac{d}{dt}\begin{bmatrix} F_c \\ v_i \end{bmatrix} = \begin{bmatrix} 0 & k \\ -1/m & 0 \end{bmatrix}\begin{bmatrix} F_c \\ v_i \end{bmatrix} \tag{4.86}$$

At first glance (comparing equation 4.86 with equation 4.74) it might appear that because the system matrix in these equations has changed, we have a different model. Not so; *we have merely expressed it in different coordinates.* Double differentiating the velocity and substituting for $\dot{F}_c$ again yields the equation of a simple harmonic oscillator.

$$\ddot{v}_i + \omega^2 v_i = 0 \tag{4.87}$$

where $\omega = \sqrt{k/m}$ as before.

In fact we can readily recover the original state equations. Denote the new state variables as $\mathbf{r}^*$.

$$\mathbf{r}^* \triangleq \begin{bmatrix} F_c \\ v_i \end{bmatrix} \tag{4.88}$$

Write the state equations in matrix notation.

$$\dot{\mathbf{r}}^* = \mathbf{A}\, \mathbf{r}^* \tag{4.89}$$

where **A** is the system matrix of equation 4.86. The new state variables, **r***, are related to the original state variables, **r**, by a constant, linear transformation matrix **M**.

$$\begin{bmatrix} F_c \\ v_i \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & 1/m \end{bmatrix} \begin{bmatrix} q_c \\ p_i \end{bmatrix} \tag{4.90}$$

or

$$\mathbf{r}^* = \mathbf{M}\,\mathbf{r} \tag{4.91}$$

The transformation matrix is non-singular, so the inverse transformation exists.

$$\begin{bmatrix} q_c \\ p_i \end{bmatrix} = \begin{bmatrix} 1/k & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} F_c \\ v_i \end{bmatrix} \tag{4.92}$$

or

$$\mathbf{r} = \mathbf{M}^{-1}\mathbf{r}^* \tag{4.93}$$

Differentiating 4.93 and substituting equations 4.89 and 4.91 we recover the equation 4.74, our original state equations.

$$\dot{\mathbf{r}} = \mathbf{M}^{-1}\dot{\mathbf{r}}^* = \mathbf{M}^{-1}\mathbf{A}\,\mathbf{r}^* = \mathbf{M}^{-1}\mathbf{A}\,\mathbf{M}\,\mathbf{r} \tag{4.94}$$

$$\frac{d}{dt}\begin{bmatrix} q_c \\ p_i \end{bmatrix} = \begin{bmatrix} 1/k & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 0 & k \\ -1/m & 0 \end{bmatrix} \begin{bmatrix} k & 0 \\ 0 & 1/m \end{bmatrix} \begin{bmatrix} q_c \\ p_i \end{bmatrix} \tag{4.95}$$

$$\frac{d}{dt}\begin{bmatrix} q_c \\ p_i \end{bmatrix} = \begin{bmatrix} 0 & 1/m \\ -k & 0 \end{bmatrix} \begin{bmatrix} q_c \\ p_i \end{bmatrix} \tag{4.74}$$

If we regard state variables as coordinates of a state space, equation 4.91 describes a (non-singular) transformation of coordinates. But as there are an infinity of non-singular constant transformation matrices, we can see why state variables, which uniquely characterize a system's state, are not themselves unique — the infinity of equivalent sets of state variables corresponds to the infinity of possible choices of coordinates. Clearly the underlying physical behavior should remain the same under a change of coordinates, though the details of the description will be different in the different coordinate frames, hence the difference between equations 4.74 and 4.86. On the other hand, experience with choice of coordinates in mathematical analysis should have taught you that for certain purposes, some coordinate choices are more convenient than others.

Because effort (force) and flow (velocity) together define the power into an element, this second choice is known as power state variables. It is a particularly convenient choice if the elements are linear, and is widely used in electric circuit theory (where the power variables are voltage and current); hence these variables are sometimes called circuit state variables.

Whichever state variables we choose, the minimum number necessary to describe energetic transactions in this system is two. Physically, there are two independent energy storing elements in the system, and the dynamic process arises from exchange of energy between them. Therefore, to define the energetic state of the system at any moment we require two quantities.

Another way to think of this: each of the two energy storage elements embodies a differential equation which is integrated to determine the solution, the state trajectory of the system. Two initial conditions will be needed to determine the constants of integration, and this is another way of saying that we require two numbers to determine the energetic state of the system.