

## Dependent Energy Storage Elements

In the foregoing examples we found that one state variable was associated with the energy stored in each energy storage element. Will every energy storage element give rise to an unique state variable? Not necessarily, as we will see below when we consider two energy storage elements of the *same* type connected by a simple junction.

Suppose we wish to model one dimension of the motion of two space vehicles in a vacuum under free-fall conditions (i.e. zero net gravitational effects). As we are only concerned with their overall motion it seems reasonable to model each vehicle as an ideal Newtonian rigid body. But then the two vehicles dock with one another such that they subsequently move with the same velocity. As the two now move with a common velocity (flow), they interact through a one junction and a bond graph for one-dimensional motion of the system would be as in figure 4.13.

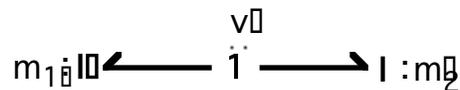


Figure 4.13: Bond graph representation of common-velocity coupling between two inertias.

Because these are ideal (linear) inertias, we may differentiate their constitutive equations as before:

$$dv_1/dt = F_1/m_1 \quad (4.96)$$

$$dv_2/dt = F_2/m_2 \quad (4.97)$$

The equations associated with the one-junction may be written as follows:

$$v_1 = v_2 \quad (4.98)$$

$$F_1 = -F_2 \quad (4.99)$$

In this case, simple substitution does not lead to state equations; a little manipulation is required.

$$dv_1/dt = -(m_2 dv_1/dt)/m_1 \quad (4.100)$$

Note this is an implicit equation. Rearranging:

$$(m_1 + m_2)dv_1/dt = 0 \quad (4.101)$$

Assuming  $m_1$  and  $m_2$  are non-zero constants,  $dv_1/dt = 0$ . Integrating, the system state trajectory is:

$$v_1(t) = \text{constant} \quad (4.102)$$

This system only requires one constant of integration, and therefore only one state variable. Yet the model had two storage elements. Why doesn't it require two state variables as in the previous example? Because the two energy storage elements in this model are *not independent*. Because of the one-junction, the velocity or momentum of one determines the velocity or momentum of the other; given the masses of both bodies, knowing the energy of one is sufficient to determine the energy of the other. Therefore only one state variable is needed; only one initial condition is required to determine the single constant of integration.

Causality and Dependent Energy Storage Elements

In previous examples, state equations were obtained by a simple process of substitution, yet in the simple example above, further algebraic manipulation was required. This is a typical consequence of dependent energy storage elements and, as one might expect, in more complex systems the algebraic manipulations can become formidable, even prohibitively so. It would be useful to know about dependent energy-storage elements before attempting to derive equations. How may we do so?

The inter-dependence of energy storage elements is easily discovered by considering causality. It refers to the choice of input and output which must be made when we come to describe a system in terms of mathematical operations<sup>1</sup> on numbers.

As we will see below, it is also closely related to cause and effect in the usual *temporal* meaning of those words, though in this context the consequences of causality may be unfamiliar. It is generally accepted that physical systems must be causal. If a cause produces an effect, then that effect may not precede the cause in time. Said another way, if a system is causal, its present output cannot depend on its future input.

*Integral Causality*

If we describe a system in terms of an input and an output then the equations used to model the system can be usefully treated as a sequence of operators which act upon an input to produce an output. Thus, in the first-order examples above, an ideal capacitor comprises two operations: an input flow is integrated to yield an output displacement; that displacement in turn determines an effort. This *integral causal form* of the capacitor equations may be represented by the operational block diagram shown in figure 4.14.

Conversely, for an inertia, an input effort may be integrated to yield an output momentum; that momentum in turn determines a flow. The integral causal form of the inertia equations may be represented by the operational block diagram shown in figure 4.15.

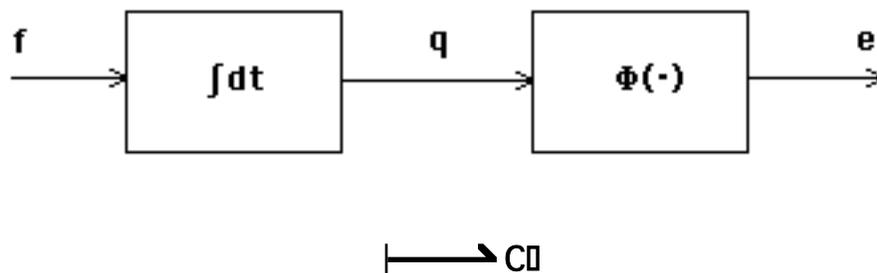


Figure 4.14: Block diagram representation of the mathematical operations associated with the integral causal form of a generalized capacitor. A corresponding bond graph is also shown.

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<sup>1</sup> For that reason it is sometimes referred to as *operational causality*.

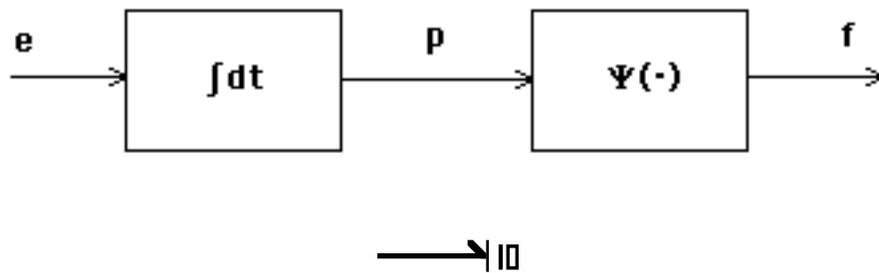


Figure 4.15: Block diagram representation of the mathematical operations associated with the integral causal form of a generalized inertia. A corresponding bond graph is also shown.

When an inertia is coupled to a capacitor through a one-junction, (as in the Hamiltonian system above) the operations required to model the complete system may be represented by combining the operational block diagrams for each of the pieces as shown in figure 4.16.

The defining equation for the one-junction (the identity of the flow variables) is simply a connection in the block diagram; The associated compatibility equation is (in this simple case) a sign change operation. This diagram shows that the complete system comprises a closed loop of operations, a chain of causes and effects. Both the inertia and the capacitor are in integral causal form and that means that all of the operations well-defined.

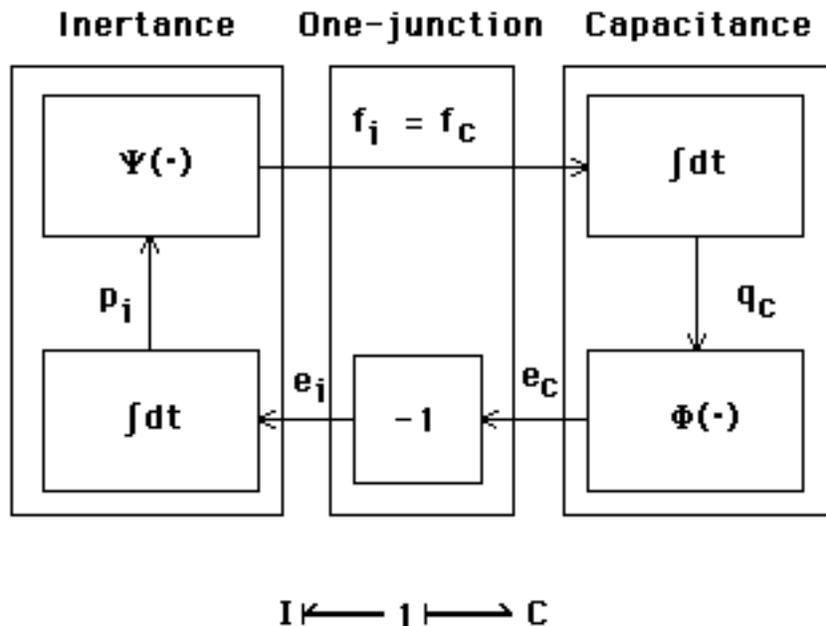


Figure 4.16: Block diagram representation of the mathematical operations associated with a capacitor and an inertia connected through a one-junction. An equivalent bond graph is also shown.

*Derivative Causality*

If we attempt to represent two inertias and a one junction (as in the example of the space vehicles) using block diagrams, we encounter a difficulty: one of the inertias has to accept flow as input and

produce effort as output, the opposite of what is shown in figure 4.15. Simply reversing the orientation of the arrows on the block diagram is meaningless; one of the two inertias (it doesn't matter which) must be represented using a different set of operations, the inverse of those depicted in figure 4.15. The constitutive equation for the inertia must be replaced by its inverse; the time-integration operator must be replaced with a time-derivative operator, as shown in figure 4.17. As in this figure, an asterisk will often be used to call attention to the time-derivative operator. An energy-storage element which is represented by a time-derivative operation is said to be in *derivative causal form*.

Figure 4.17 shows that, as before, the complete system comprises a closed loop of operations, a chain of causes and effects. However, unlike figure 4.16, the derivative causality of one of the inertias in figure 4.17 means that all of the operations in the loop may not be well-defined. First, the inverse of the constitutive equation for the inertia may not exist. In the example above it does, as the constitutive equation is linear, but in general a well-defined function may not have a well-defined inverse. A second and more profound problem stems from the time-derivative operation; it is highly undesirable because *time differentiation is a physically impossible operation*.

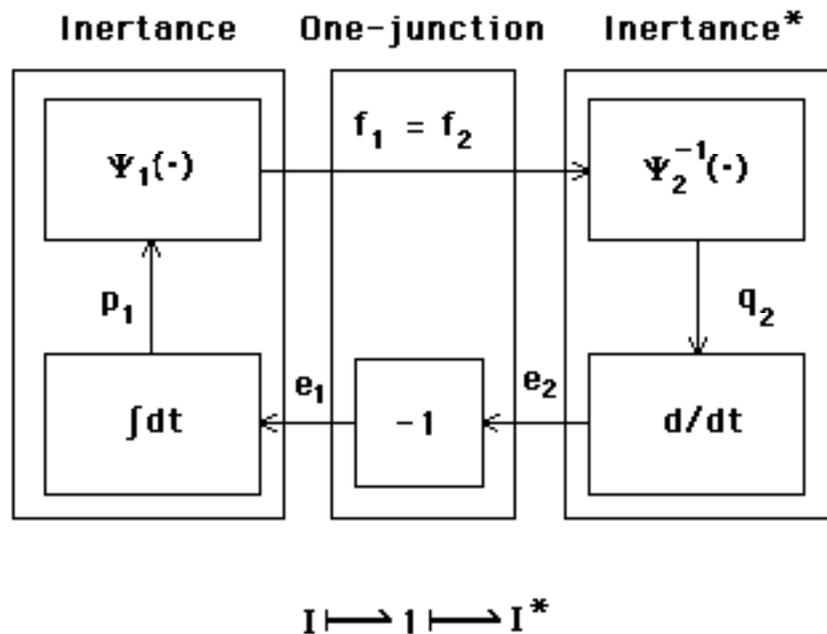


Figure 4.17: Block diagram representation of the mathematical operations associated with two inertias connected through a one-junction. An equivalent bond graph is also shown.

There are several ways to understand this. The most fundamental is based on the definition of a time derivative as the limit of the difference of two values of a function divided by their time separation as the latter goes toward zero. To obtain the time derivative of a function at any point in time one must take the difference of two values of the function, one in the immediate past, *and one in the immediate future*. This is illustrated in figure 4.18. The output of a time derivative operation depends on future values of its input, and therefore it is non-causal in the usual physical sense.

An alternative argument: The input of any operational block may be any well-defined function of time, including a discontinuous function such as a step function. For the integration operator, discontinuities present no problem: the input force applied to an inertial object may change abruptly; its velocity, momentum and stored kinetic energy will not.

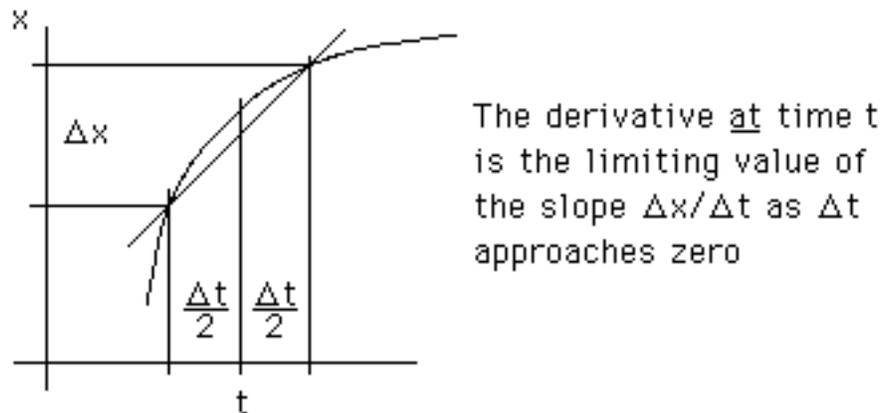


Figure 4.18: Diagram illustrating the fundamental definition of a derivative operation.

In contrast, the derivative operator will generate infinite output in response to a discontinuous input, and this infinity is a departure from the physically realizable. A discontinuous change in the velocity of an inertial object would require an infinite force. Furthermore, the kinetic energy stored in the inertial object would also change discontinuously, and that would require an infinite power flow. Neither of these infinities is physically possible. For these reasons, a time differentiation operation cannot be physically realized; it can at best be approximated.

How much does this matter? We know from thermodynamics that real macroscopic phenomena are not reversible. The time sequencing of events is a fundamental aspect of physical system behavior. Therefore causality is an important aspect of our description of macroscopic physical phenomena. The point is that to use a time derivative operation in a model of a physical system is a major misrepresentation of physical reality as we understand it and it should be avoided.

#### *Derivative Causality Indicates Dependent Energy Storage*

However, to put this discussion in perspective, every mathematical model falls short of reality, usually very far short; for example, the idea of a rigid body (which we used in the example with the two space vehicles) is itself a fiction, albeit a very convenient one — a typical space vehicle is anything but rigid. It can then be argued that a time-derivative operation is intrinsically no worse than this and other modelling approximations.

Indeed, when two inertias are coupled by a one-junction, *the resulting derivative causality really means that the two inertias are one object*, one rigid body. That is the true meaning of interdependence of energy storage elements: in the model they are not distinct energy storage elements, despite appearances to the contrary. These two modelling approximations — rigid-body models and time-derivative operations — are intimately related.

If we consider an object undergoing translational motion along one dimension in space, the assumption that it is rigid is equivalent to assuming that the (infinitely) large collection of fictional "mass points" which comprise the object all move with a common speed. In bond-graph notation, if each mass point were represented by an inertia, all would be connected to a single one-junction representing the common speed. But in terms of mathematical operations, only one of those inertias can determine the common speed associated with the one-junction; only one inertia can have integral causality. All the others must accept that speed as an input; they must be in derivative causality. The entire collection of mass points is a single independent energy storage element; a

single number (the common momentum or common speed) is sufficient to determine the stored energy.

*Preferred Causality for Energy Storage Elements*

A point to be taken from this discussion is that, if possible, energy-storage elements should be independent and have integral causality. But why? Why does it matter if an energy-storage element is dependent? If it is regarded as merely another modelling approximation, why should the time-derivative operation be avoided?

Heed the admonition of the the fourteenth-century scholar William of Ockham<sup>2</sup>:

"Entia praeter necessitatem non sunt multiplicanda"

In the modelling context: Approximations must not be made needlessly; wherever possible, the integration operation should be used rather than the derivative operation.

Every energy-storage element which can be described using an integration operator should be. It will require one initial condition to determine its constant of integration, and therefore will give rise to one state variable; energy storage elements which have integral causality are independent. Conversely, any energy storage element which must be described using a derivative operation will not require an independent initial condition and therefore will not give rise to a state variable; energy storage elements which have derivative causality are dependent.

A final comment: Elements such as power sources are subject to *causal constraints*; their definition will only permit one causal form. In contrast, in certain situations which we will discuss later, derivative causality is unavoidable, or the steps required to eliminate it would obscure the insight that the model is supposed to provide. The integral causal form is the *preferred* causal form for energy-storage elements; it is *not* essential.

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<sup>2</sup> If I have my Latin correct, this translates as "Entities are not to be multiplied beyond necessity"; it is popularly known as Occam's Razor.