Ideal asymmetric junction elements

Relax the symmetry assumption and examine the resulting junction structure. For simplicity, consider two-port junction elements.

As before, assume instantaneous power transmission between the ports without storage or dissipation of energy. Characterize the power flow in and out of a two-port junction structure using four real-valued wave-scattering variables. Using vector notation:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \text{(A.1)}$$

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \text{(A.2)}$$

The input and output power flows are the square of the length of these vectors, their inner products.

$$P_{in} = \sum_{i=1}^{2} u_i^2 = \mathbf{u}^t\mathbf{u} \qquad \text{(A.3)}$$

$$P_{out} = \sum_{i=1}^{2} v_i^2 = \mathbf{v}^t\mathbf{v} \qquad \text{(A.4)}$$

The constitutive equations of the junction structure may be written as follows.

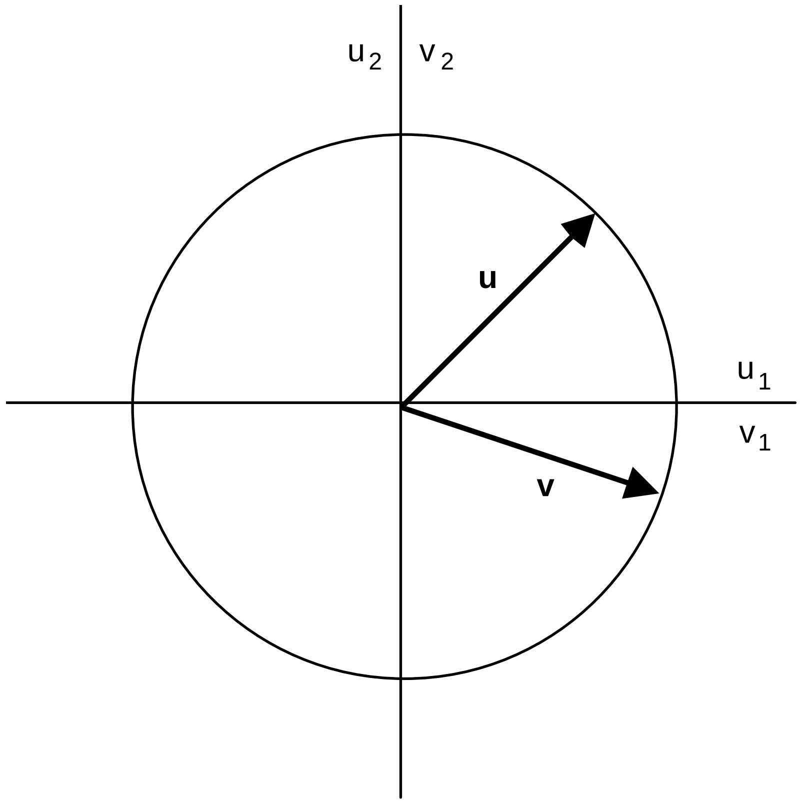
$$\mathbf{v} = \mathbf{f}(\mathbf{u}) \qquad \text{(A.5)}$$

Geometrically, the requirement that power in equal power out means that the length of the vector **v** must equal the length of the vector **u**, i.e. their tips must lie on the perimeter of a circle (see figure A.1).

For any two particular values of **u** and **v**, the algebraic relation $\mathbf{f}(\cdot)$ is equivalent to a *rotation operator*.

$$\mathbf{v} = \mathbf{S}(\mathbf{u})\,\mathbf{u} \qquad \text{(A.6)}$$

where the square matrix **S** is known as a *scattering matrix*.



S need not be a constant matrix, but may in general depend on the power flux through the junction, hence the notation **S**(**u**). However, **S** is subject to important restrictions. In particular,

$$\mathbf{v}^t\mathbf{v} = \mathbf{u}^t\mathbf{S}^t\mathbf{S}\mathbf{u} = \mathbf{u}^t\mathbf{u} \qquad \text{(A.7)}$$

S is ortho-normal matrix: the vectors formed by each of its rows (or columns) are (i) orthogonal and (ii) have unit magnitude; its transpose is its inverse.

$$\mathbf{S}^t\mathbf{S} = \mathbf{1} \qquad \text{(A.8)}$$

This constrains the coefficients of the scattering matrix as follows.

$$\mathbf{S} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{(A.9)}$$

$$a^2 + c^2 = 1 \qquad \text{(A.10)}$$

$$ab + cd = 0 \qquad \text{(A.11)}$$

$$b^2 + d^2 = 1 \qquad \text{(A.12)}$$

As there are only three independent equations and four unknown quantities, we see that this junction is characterized by a single parameter. We may also write the orthogonality condition as

$$\mathbf{S}\mathbf{S}^t = \mathbf{1} \qquad \text{(A.13)}$$

which yields the following equations.

$$a^2 + b^2 = 1 \qquad \text{(A.14)}$$

$$ac + bd = 0 \qquad \text{(A.15)}$$

$$c^2 + d^2 = 1 \qquad \text{(A.16)}$$

There are four possible solutions to these equations. Combining A.10 and A.16,

$a^2 = 1 - c^2 = d^2$. Thus $a = \pm d$.

If $a = d$ then $b = c = \pm\sqrt{1 - a^2}$.

<u>One solution</u>

Choosing the positive root yields one solution. Assuming the coefficient a to be the undetermined parameter,

$$\mathbf{S} = \begin{bmatrix} a & \sqrt{1 - a^2} \\ -\sqrt{1 - a^2} & a \end{bmatrix} \qquad \text{(A.17)}$$

Rewrite in terms of effort and flow variables.

$$\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \qquad \text{(A.18)}$$

$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \qquad \text{(A.19)}$$

The relation between efforts and wave-scattering variables is as follows.

$$\mathbf{e} = (\mathbf{u} - \mathbf{v})\, c = c\,(\mathbf{1} - \mathbf{S})\,\mathbf{u} \qquad \text{(A.20)}$$

where c is a scaling constant. The relation between flows and wave-scattering variables is as follows.

$$\mathbf{f} = (\mathbf{u} + \mathbf{v})/c = 1/c\ (\mathbf{1} + \mathbf{S})\ \mathbf{u} \quad \text{(A.21)}$$

If $|a| \neq 1$ then $\mathbf{1} + \mathbf{S}$ and $\mathbf{1} - \mathbf{S}$ are nonsingular matrices and the input wave scattering variables u_1 and u_2 may be eliminated as follows.

$$\mathbf{e} = c^2\ (\mathbf{1} - \mathbf{S})\ (\mathbf{1} + \mathbf{S})^{-1}\ \mathbf{f} \quad \text{(A.22)}$$

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = c^2 \begin{bmatrix} 0 & -\sqrt{(1 - a)/(1 + a)} \\ \sqrt{(1 - a)/(1 + a)} & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \quad \text{(A.23)}$$

Writing $G = c^2\sqrt{(1 - a)/(1 + a)}$ we obtain the equation for an <u>ideal gyrator</u>.

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 & -G \\ G & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \quad \text{(A.24)}$$

Note that equations A.20 and A.21 imply a sign convention in effort-flow coordinates such that power is positive inwards on both ports.

$$P_{\text{net inwards}} = \mathbf{e}^t\mathbf{f} = \mathbf{u}^t\mathbf{u} - \mathbf{v}^t\mathbf{v} \quad \text{(A.25)}$$

To follow the more common sign convention we may simply change the sign of f_2 in equation A.24.

If $a = 1$, $\mathbf{e}$ is identically zero for all values of $\mathbf{f}$. No energy is exchanged between the ports and the junction structure behaves like a dissipator with zero resistance.

If $a = -1$, $\mathbf{f}$ is identically zero for all values of $\mathbf{e}$. No energy is exchanged between the ports and the junction structure behaves like a dissipator with infinite resistance (zero conductance).

A second solution

Choosing a = d and using the negative root yields another solution. Again assuming the coefficient a to be the undetermined parameter,

$$\mathbf{S} = \begin{bmatrix} a & -\sqrt{1 - a^2} \\ \sqrt{1 - a^2} & a \end{bmatrix} \tag{A.26}$$

In this case the relation between efforts and flows is

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = c^2 \begin{bmatrix} 0 & \sqrt{(1 - a)/(1 + a)} \\ -\sqrt{(1 - a)/(1 + a)} & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \tag{A.27}$$

Again we obtain the equation for an ideal gyrator.

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 & G \\ -G & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \tag{A.28}$$

A third solution

If $a = -d$, $b = c = \pm\sqrt{1 - a^2}$. Using the positive root and assuming a to be the undetermined parameter

$$\mathbf{S} = \begin{bmatrix} a & \sqrt{1 - a^2} \\ \sqrt{1 - a^2} & -a \end{bmatrix} \tag{A.29}$$

In this case the matrices **1** + **S** and **1** - **S** are singular for all values of the parameter a.

However, equations A.20 and A.21 may be combined as follows:

$$\begin{bmatrix} e_1/c \\ e_2/c \\ cf_1 \\ cf_2 \end{bmatrix} = \begin{bmatrix} \mathbf{1} - \mathbf{S} \\ \text{-----} \\ \mathbf{1} + \mathbf{S} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{A.30}$$

$$\begin{bmatrix} e_1/c \\ e_2/c \\ cf_1 \\ cf_2 \end{bmatrix} = \begin{bmatrix} 1 - a & -\sqrt{1 - a^2} \\ -\sqrt{1 - a^2} & 1 + a \\ 1 + a & \sqrt{1 - a^2} \\ \sqrt{1 - a^2} & 1 - a \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{A.31}$$

If $|a| \neq 1$, the 4 x 2 matrix relating efforts and flows to the input scattering variables contains two nonsingular 2 x 2 submatrices.

$$\begin{bmatrix} e_2/c \\ cf_1 \end{bmatrix} = \begin{bmatrix} -\sqrt{1 - a^2} & 1 + a \\ 1 + a & \sqrt{1 - a^2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{A.32}$$

$$\begin{bmatrix} e_1/c \\ cf_2 \end{bmatrix} = \begin{bmatrix} 1 - a & -\sqrt{1 - a^2} \\ \sqrt{1 - a^2} & 1 - a \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{A.33}$$

Solving the second of these for **u** and substituting into the first we obtain a relation between efforts and flows.

$$\begin{bmatrix} e_2 \\ f_1 \end{bmatrix} = \begin{bmatrix} -\sqrt{(1 + a)/(1 - a)} & 0 \\ 0 & \sqrt{(1 + a)/(1 - a)} \end{bmatrix} \begin{bmatrix} e_1 \\ f_2 \end{bmatrix} \qquad \text{(A.34)}$$

Writing $T = \sqrt{(1 + a)/(1 - a)}$ we obtain the equation for an <u>ideal transformer</u>.

$$\begin{bmatrix} e_2 \\ f_1 \end{bmatrix} = \begin{bmatrix} -T & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} e_1 \\ f_2 \end{bmatrix} \qquad \text{(A.35)}$$

To follow the more common sign convention we may change the sign of e_2.

If the parameter $a = \pm 1$, an argument similar to that used above shows that a degenerate case results in which no energy is exchanged between the ports.

Final solution

Choosing a = d and using the negative root we obtain the fourth solution.

$$\mathbf{S} = \begin{bmatrix} a & -\sqrt{1 - a^2} \\ -\sqrt{1 - a^2} & -a \end{bmatrix} \tag{A.36}$$

Once again, the matrices **1** + **S** and **1** - **S** are singular for all values of the parameter a, but by rearranging equations A.20 and A.21 as before the corresponding relation between efforts and flows is

$$\begin{bmatrix} e_2 \\ f_1 \end{bmatrix} = \begin{bmatrix} \sqrt{(1 + a)/(1 - a)} & 0 \\ 0 & -\sqrt{(1 + a)/(1 - a)} \end{bmatrix} \begin{bmatrix} e_1 \\ f_2 \end{bmatrix} \tag{A.37}$$

Again we obtain the equation for an ideal transformer

$$\begin{bmatrix} e_2 \\ f_1 \end{bmatrix} = \begin{bmatrix} T & 0 \\ 0 & -T \end{bmatrix} \begin{bmatrix} e_1 \\ f_2 \end{bmatrix} \tag{A.38}$$

Two-port junction elements

There are only two possible power-continuous, asymmetric two-port junction elements, the gyrator and the transformer.

Unlike the ideal symmetric junction elements (**0** and **1**) the ideal asymmetric junction elements may be nonlinear.

The relation between efforts and flows must have a *multiplicative* form.

The general asymmetric junction elements are a modulated gyrator (**MGY**) and a modulated transformer (**MTF**) respectively.