CO-ENERGY (AGAIN)

In the linear case, energy and co-energy are numerically equal.

—the value of distinguishing between them may not be obvious.

Why bother with co-energy at all?

EXAMPLE: SOLENOID WITH MAGNETIC SATURATION.

Previous solenoid constitutive equations assumed electromagnetic linearity.

—arbitrarily large magnetic fluxes could be generated.

In reality flux cannot exceed saturation flux.

For sufficiently high currents behavior is strongly nonlinear.

MODEL THAT PHENOMENON.

Assume an electrical constitutive equation as follows:

$$\lambda(i, x) = \frac{L(x)\, i\, \lambda_S}{\sqrt{L(x)^2 i^2 + \lambda_S^2}}$$

where L(x) is position-dependent inductance as before.

For sufficiently small currents the relation is approximately linear.

$$i << \frac{\lambda_S}{L(x)} \qquad \lambda \approx L(x)\, i$$

For sufficiently large currents the flux linkage reaches a limiting value, λ_S.

$$i >> \frac{\lambda_S}{L(x)} \qquad \lambda \approx \lambda_S$$

MECHANICAL CONSTITUTIVE EQUATION

may be found using the stored energy.

- **Find the stored energy at a fixed displacement.**
- **Find force as the gradient with respect to displacement.**

That yields the relation between force and flux linkage.

$$F = F(\lambda, x)$$

But flux cannot be specified arbitrarily.

Realistic boundary conditions require current input.

To find the relation between force and current, substitute.

$$\lambda = \lambda(i, x)$$

STORED ELECTRICAL ENERGY

(at a fixed displacement)

$$E = \int i \, d\lambda$$

Need to invert the relation between flux linkage and current.

—In general, anything but straightforward.

In this case, a little algebra yields the following.

$$i = \frac{\lambda_S}{L(x)} \frac{\lambda}{\sqrt{\lambda_S^2 - \lambda^2}} \qquad |\lambda| \leq |\lambda_S|$$

USE WITH CAUTION!

If $\lambda > \lambda_S$ this expression yields an imaginary number for the current.

A LITTLE CALCULUS

(and some more algebra) yields an expression for energy.

$$E(\lambda, x) = \frac{\lambda_s^2}{L(x)}\left(1 - \sqrt{1 - \left(\frac{\lambda}{\lambda_s}\right)^2}\right)$$

see Note 1 attached

Partial derivative with respect to displacement:

$$F(\lambda, x) = -\frac{\partial L(x)}{\partial x}\frac{\lambda_s^2}{L(x)^2}\left(1 - \sqrt{1 - \left(\frac{\lambda}{\lambda_s}\right)^2}\right)$$

Substitute for flux linkage:

$$F(i, x) = -\frac{\partial L(x)}{\partial x}\frac{\lambda_s^2}{L(x)^2}\left(1 - \sqrt{1 - \left(\frac{L(x)i}{\sqrt{L(x)^2 i^2 + \lambda_S^2}}\right)^2}\right)$$

Still more algebra simplifies this expression:

$$F(i, x) = -\frac{\partial L(x)}{\partial x}\frac{\lambda_s^2}{L(x)^2}\left(1 - \frac{\lambda_S}{\sqrt{L(x)^2 i^2 + \lambda_S^2}}\right)$$

COMMENT:

The assumed electrical constitutive equation

$$\lambda(i, x) = \frac{L(x) i \lambda_s}{\sqrt{L(x)^2 i^2 + \lambda_s^2}}$$

was chosen primarily for pedagogic simplicity

- **—a more realistic relation may be (far) less tractable**
- **—usually no simple algebraic form exists**

Consequently a procedure requiring

(a) inversion

(b) integration with respect to flux linkage and

(c) differentiation with respect to displacement

may be impractical.

AN ALTERNATIVE APPROACH:

use co-energy instead of energy.

Total stored energy:

$$E = \int i \, d\lambda + \int F \, dx$$

Electrical co-energy:

$$E^* = i\lambda - E = i\lambda - \int i \, d\lambda - \int F \, dx$$

(a Legendre transformation with respect to current)

Force is the *negative* gradient of this co-energy.

$$\frac{\partial E^*}{\partial x} = -\frac{\partial E}{\partial x} = -F$$

Electrical co-energy at a fixed displacement:

$$E^* = \int \lambda \; di$$

$$E^*(i, x) = \frac{\lambda_s}{L(x)}\left(\sqrt{L(x)^2 i^2 + \lambda_s^2} - \lambda_s\right)$$

see Note 2 attached

KEY POINT:

Co-energy may be found without inverting the relation between flux linkage and current.

Partial differentiation with respect to displacement:

$$F(i, x) = -\frac{\partial L(x)}{\partial x}\frac{\lambda_s^2}{L(x)^2}\left(1 - \frac{\lambda_s}{\sqrt{L(x)^2 i^2 + \lambda_s^2}}\right)$$

see Note 3 attached

No further algebra required.

REMARKS

In general, co-energy functions are useful for multiport and/or nonlinear energy storage elements.

Inverting constitutive equations may be avoided.

In the linear case, energy and co-energy are numerically equal.

the value of distinguishing between them may not be obvious.

In the nonlinear case, the two are not equal.

Distinguishing between them is important.

In this example,

energy is upper-bounded

co-energy is not.

In general,

energy is conserved

co-energy need not be.

NOTE 1:

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Integration to find energy:

$$E(\lambda, x) = \int_0^{\lambda} \frac{\lambda_s}{L(x)} \frac{y\,dy}{\sqrt{\lambda_s^2 - y^2}} \qquad |\lambda| \leq |\lambda_s|$$

where y is a "dummy variable".

rearrange:

$$E(\lambda, x) = \int_0^{\lambda} \frac{1}{L(x)} \frac{y\,dy}{\sqrt{1 - \left(\frac{y}{\lambda_s}\right)^2}}$$

substitution: define

$$\sin u = \frac{y}{\lambda_S}$$

$$\sqrt{1 - \left(\frac{y}{\lambda_S}\right)^2} = \sqrt{1 - \sin^2 u} = \sqrt{\cos^2 u} = \cos u$$

$$y = \lambda_S \sin u$$

$$u = \sin^{-1}\left(\frac{y}{\lambda_S}\right) \quad -\pi/2 \leq u \leq \pi/2$$

$$dy = \lambda_S \cos u \, du \quad \text{(assuming } dx = 0\text{)}$$

$$E(\lambda, x) = \int_0^{\sin^{-1}(\lambda/\lambda_S)} \frac{1}{L(x)} \frac{\lambda_S \sin u}{\cos u} \lambda_S \cos u \, du$$

$$E(\lambda, x) = \int_0^{\sin^{-1}(\lambda/\lambda_S)} \frac{\lambda_S^2}{L(x)} \sin u \, du$$

$$E(\lambda, x) = \int_0^{\sin^{-1}(\lambda/\lambda_S)} \frac{\lambda_S^2}{L(x)}\, d(-\cos u)$$

$$E(\lambda, x) = -\frac{\lambda_S^2}{L(x)}\left(\cos\left(\sin^{-1}\left(\frac{\lambda}{\lambda_S}\right)\right) - \cos(0)\right)$$

$$\cos\left(\sin^{-1}\left(\frac{\lambda}{\lambda_S}\right)\right) = \sqrt{1 - \left(\frac{\lambda}{\lambda_S}\right)^2}$$

$$E(\lambda, x) = \frac{\lambda_S^2}{L(x)}\left(1 - \sqrt{1 - \left(\frac{\lambda}{\lambda_S}\right)^2}\right)$$

Does that make sense?

In the limit as λ approaches λ_S

$$E(\lambda, x) \approx \frac{\lambda_S^2}{L(x)}$$

Plausible; implies that only a finite amount of energy may be stored.

In the limit as λ approaches 0, binomial series expansion of the root yields

$$\sqrt{1 - \left(\frac{\lambda}{\lambda s}\right)^2} = 1 - \frac{1}{2}\left(\frac{\lambda}{\lambda s}\right)^2 + \text{higher order terms}$$

$$E(\lambda, x) \approx \frac{\lambda s^2}{L(x)}\left(1 - 1 + \frac{1}{2}\left(\frac{\lambda}{\lambda s}\right)^2\right)$$

$$E(\lambda, x) \approx \frac{\lambda^2}{2\,L(x)}$$

as expected for a linear inductor.

NOTE 2:

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Integration to find co-energy:

$$E^*(i, x) = \int_0^i \frac{L(x)\, i\, \lambda_S}{\sqrt{L(x)^2 i^2 + \lambda_S^2}}\, di$$

$$d\left(\sqrt{L(x)^2 i^2 + \lambda_S^2}\right) = \frac{L(x)^2\, i\, di}{\sqrt{L(x)^2 i^2 + \lambda_S^2}}$$

$$E^*(i, x) = \frac{\lambda_S}{L(x)} \int_0^i d\left(\sqrt{L(x)^2 i^2 + \lambda_S^2}\right)$$

$$E^*(i, x) = \frac{\lambda_S}{L(x)} \left(\sqrt{L(x)^2 i^2 + \lambda_S^2} - \sqrt{\lambda_S^2}\right)$$

Does this make sense?

In the limit as

$$L(x)\, i \gg \lambda_S \quad E^* \approx \lambda_S\, i$$

This is the area of a rectangle of sides i and λ_S. Note that co-energy may increase without bound, whereas energy may not.

In the limit as

$$L(x)\, i \ll \lambda_S$$

series expansion of the square root

$$\sqrt{L(x)^2 i^2 + \lambda_S^2} = (\lambda_S^2)^{1/2} + \frac{1}{2} (\lambda_S^2)^{-1/2} L(x)^2 i^2 + \text{higher order terms}$$

$$E^*(i, x) \approx \frac{\lambda_S}{L(x)} \left((\lambda_S^2)^{1/2} + \frac{1}{2} (\lambda_S^2)^{-1/2} L(x)^2 i^2 - (\lambda_S^2)^{1/2} \right)$$

$$E^*(i, x) \approx \frac{1}{2}\, L(x)\, i^2$$

as expected for a linear inductor.

NOTE 3:

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Partial differentiation with respect to displacement:

$$F(i,x) = -\frac{\partial E^*}{\partial x}$$

$$\frac{\partial E^*}{\partial x} = -\frac{\partial L(x)}{\partial x}\frac{\lambda_s}{L(x)^2}\left(\sqrt{L(x)^2 i^2 + \lambda_s^2} - \lambda_s\right) + \frac{\lambda_s}{L(x)}\left(\frac{L(x) i^2}{\sqrt{L(x)^2 i^2 + \lambda_s^2}}\right)\frac{\partial L(x)}{\partial x}$$

$$\frac{\partial E^*}{\partial x} = -\frac{\partial L(x)}{\partial x}\frac{\lambda_s}{L(x)^2}\left(\frac{L(x)^2 i^2 + \lambda_s^2 - \lambda_s\sqrt{L(x)^2 i^2 + \lambda_s^2} - L(x)^2 i^2}{\sqrt{L(x)^2 i^2 + \lambda_s^2}}\right)$$

$$F(i,x) = -\frac{\partial L(x)}{\partial x}\frac{\lambda_s^2}{L(x)^2}\left(1 - \frac{\lambda_s}{\sqrt{L(x)^2 i^2 + \lambda_s^2}}\right)$$

An expanded form may be easier to understand physically:

$$F(i,x) = -\frac{\partial L(x)}{\partial x}\frac{\lambda_s^2}{L(x)^2}\left(\frac{L(x)^2 i^2 + \lambda_s^2 - \lambda_s\sqrt{L(x)^2 i^2 + \lambda_s^2}}{L(x)^2 i^2 + \lambda_s^2}\right)$$