

NONLINEAR MECHANICAL SYSTEMS (MECHANISMS)

The analogy between dynamic behavior in different energy domains can be useful.

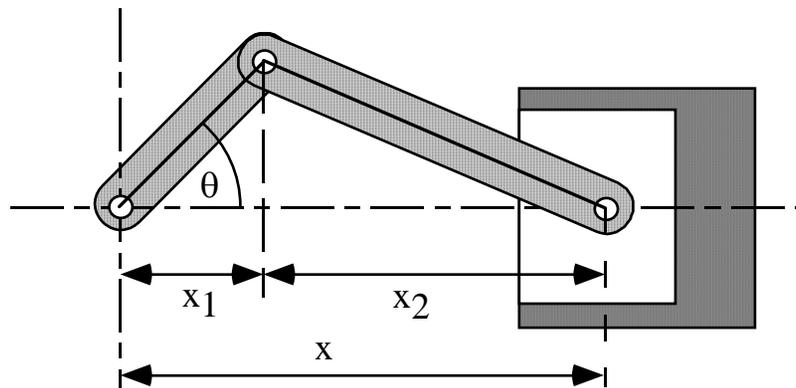
Closer inspection reveals that the analogy is not complete.

One key distinction of mechanical systems is the role of kinematics
– the geometry of motion

EXAMPLE:

automobile internal combustion engine.

reciprocating translational motion of a piston is converted to crankshaft rotation by a crank and slider mechanism.



Consider torque applied to rotate the crankshaft.

Inertia and friction opposing crankshaft rotation depend on position.

if $\theta \approx 0^\circ$ or $\theta \approx 180^\circ$ small crankshaft rotation does not move the piston

– inertia is small

if $\theta \approx \pm 90^\circ$ crankshaft rotation is nearly proportional to piston translation

– inertia is large

How should this phenomenon be modeled?

CAUTION:

A POSITION-MODULATED INERTIA WOULD VIOLATE ENERGY CONSERVATION!

$$S_e \xrightarrow[\omega]{\tau} I \leftarrow \theta$$

SHOULD THIS BE SOME KIND OF MULTIPORT?

unclear

**the angle that "modulates" inertia also modulates stored kinetic energy
that change of stored energy should be associated with a power port**

**... but that angle is already associated with a power port
how can a separate port be justified?**

EXAMINE THE MECHANISM KINEMATICS

The mechanism imposes a relation between crankshaft angle and piston position

$$x = \sqrt{l_2^2 - l_1^2 \sin^2(\theta)} + l_1 \cos(\theta)$$

where

l_1 = crank length

l_2 = connecting rod length

check:

$$\theta = 0^\circ \quad x = l_2 + l_1$$

$$\theta = \pm 90^\circ \quad x = \sqrt{l_2^2 - l_1^2}$$

$$\theta = 180^\circ \quad x = l_2 - l_1$$

OK

MODULATED TRANSFORMER

The kinematic relation

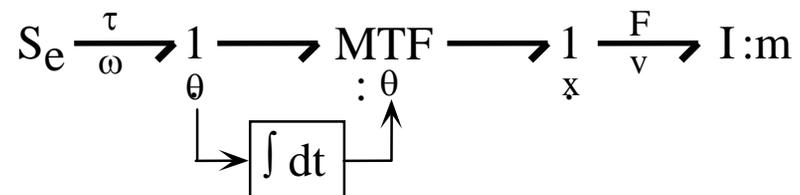
$$x = x(\theta)$$

may be regarded as a transformation between coordinates.

An energetically consistent description of the mechanism inertia includes a *modulated transformer*.

For example,

if the inertia of the crank and connecting rod are neglected, and the piston mass is assumed to be the dominant inertia, a bond graph of that model is



The angle-dependent transformer modulus is found by differentiating the kinematic relation

$$v = \frac{dx}{dt} = \left(\frac{dx}{d\theta} \right) \left(\frac{d\theta}{dt} \right) = j(\theta) \omega$$

$j(\theta)$ is often called the *Jacobian* of the transformation $x(\theta)$

For the crank and slider mechanism:

$$\frac{dx}{dt} = \left(\frac{-l_1^2 \sin(\theta) \cos(\theta)}{\sqrt{l_2^2 - l_1^2 \sin^2(\theta)}} - l_1 \sin(\theta) \right) \frac{d\theta}{dt}$$

$$j(\theta) = \left(\frac{-l_1^2 \sin(\theta) \cos(\theta)}{\sqrt{l_2^2 - l_1^2 \sin^2(\theta)}} - l_1 \sin(\theta) \right)$$

The modulated transformer also relates crankshaft torque to piston force

$$\tau = j(\theta) F$$

such that power in = power out

$$\tau \omega = j(\theta) F \omega = F j(\theta) \omega = F v$$

Thus the Jacobian is an angle-dependent moment arm.

check:

$$\theta = 0^\circ \quad j = 0$$

$$\theta = 90^\circ \quad j = -l_1$$

$$\theta = 180^\circ \quad j = 0$$

$$\theta = 270^\circ \quad j = l_1$$

OK

CAUTION:

there are subtleties to modulated transformers

- 1. THE TRANSFORMER RELATIONS MAY BE PROPERLY DEFINED IN ONLY ONE DIRECTION.**

The map $\theta \Rightarrow x$ is well defined

- there is an unique x for every θ**

The map $x \Rightarrow \theta$ is poorly defined

- for some values of x the relation does not exist**

When the relation exists it is ambiguous

- many values of θ correspond to the same x**

NOTE:

if the relation between displacements and flows is well defined in one direction

then the relation between efforts and momenta* is well defined in the other

e.g., $v = j(\theta) \omega$ is well defined for all θ

but $j(\theta) = 0$ at $\theta = 0^\circ$ and $\theta = 180^\circ$

thus $\omega = j^{-1}(\theta) v$ is not defined at these singular points

conversely, $\tau = j(\theta) F$ is well defined for all θ

but $F = j^{-1}(\theta) \tau$ is not defined at the singular points

***discussed later**

2. THE MINIMUM NUMBER OF STATE VARIABLES TO DESCRIBE SYSTEM ENERGY MAY BE UNCLEAR

Causal analysis reveals a single independent energy storage element

– this suggests the minimum system order is one

but an additional state equation is required to compute the angle

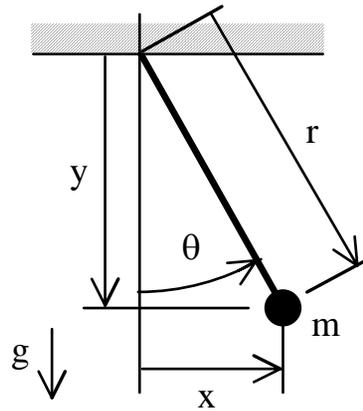
– in fact the minimum system order is two

Furthermore, this causal form requires the inverse map (or the inverse jacobian) to find the angle.

3. ACTIVE ELEMENTS MAY APPEAR AS PASSIVE ELEMENTS

A kinematically constrained source may behave like a storage elements
– but significant differences remain.

EXAMPLE: SIMPLE PENDULUM



develop a network model of this system

A general procedure to find network models for mechanisms:

1. IDENTIFY GENERALIZED COORDINATE(S)

**– a set of variables that uniquely define system configuration
in this case, angle θ may be used as a generalized coordinate**

**2. IDENTIFY THE KINEMATIC RELATION(S) DEFINING OTHER RELEVANT
DISPLACEMENTS (COORDINATES)**

**in this example, the vertical position, y , of the mass is relevant
kinematics:**

$$y = r \cos(\theta)$$

3. DIFFERENTIATE THE KINEMATIC RELATION(S) TO FIND THE TRANSFORMER MODULUS

$$\frac{dy}{dt} = -r \sin(\theta) \frac{d\theta}{dt}$$

velocities:

$$v = \frac{dy}{dt}$$

$$\omega = \frac{d\theta}{dt}$$

Jacobian:

$$j(\theta) = -r \sin(\theta)$$

4. THE BASIC JUNCTION STRUCTURE IS

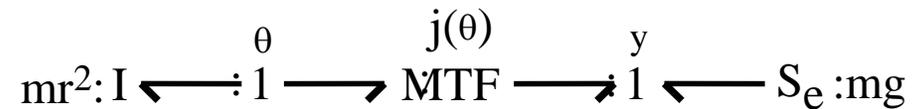


Other junction elements may be added as needed.

5. ADD ENERGY STORAGE, DISSIPATION AND SOURCE ELEMENTS.

in this case kinetic energy storage (a rotational inertia) is associated with angular speed

gravity may be described as a constant-force source that does work when vertical position changes



Causal analysis shows only one independent energy storage element – the inertia.

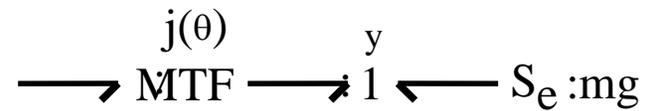
– that suggests the minimum system order is one.

From the previous argument we know that an additional state variable is needed to compute the angle.

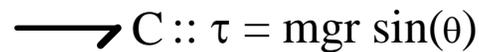
– the minimum system order is two.

But in addition, this system can oscillate.

The combination of effort source and modulated transformer can behave like a mechanical spring.



is equivalent to



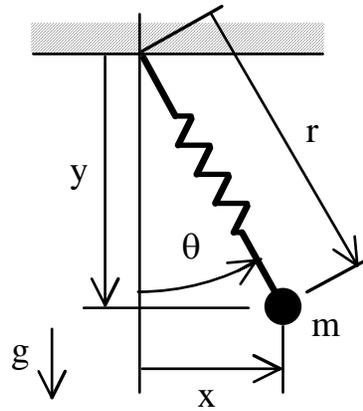
The resulting graph indicates the system can oscillate.



However, this “spring” has a “hidden” causal constraint
 – effort (torque) must be the output variable.

EXAMPLE: SPRING PENDULUM

THE PROCEDURE APPLIES TO SYSTEMS WITH MANY GENERALIZED COORDINATES



Find a network model for this mechanical system

1. IDENTIFY GENERALIZED COORDINATES

polar coordinates, r and θ , are one suitable choice

2. IDENTIFY THE KINEMATIC RELATIONS DEFINING OTHER RELEVANT DISPLACEMENTS (COORDINATES)

Cartesian coordinates, x and y , are relevant

$$x = r \sin(\theta)$$

$$y = r \cos(\theta)$$

Note:

By the definition of generalized coordinates, these relations will always be well-defined.

3. DIFFERENTIATE THE KINEMATIC RELATION(S) TO FIND THE TRANSFORMER MODULI

Velocities:

$$u = dx/dt$$

$$v = dy/dt$$

$$n = dr/dt$$

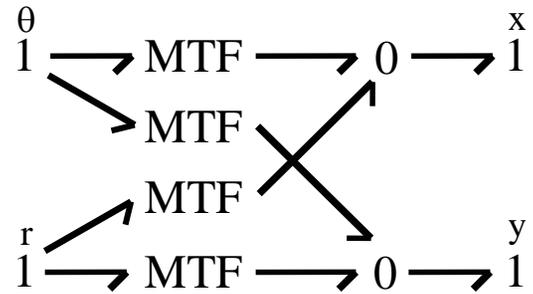
$$\omega = d\theta/dt$$

Jacobian:

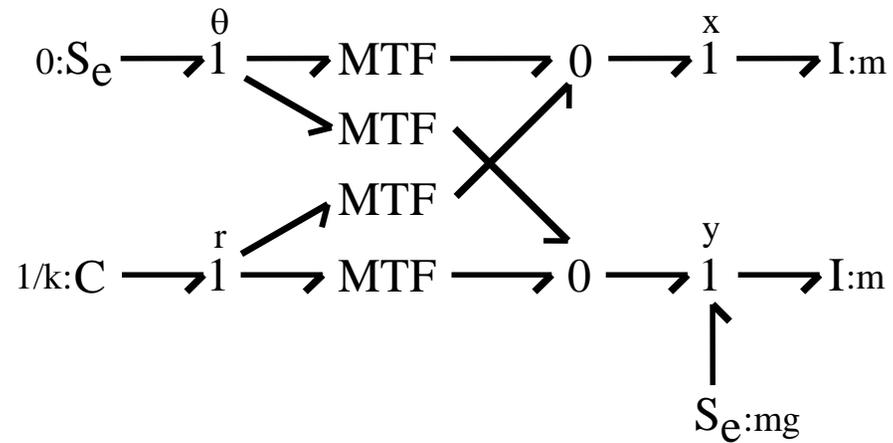
$$u = n \sin(\theta) + r \cos(\theta) \omega$$

$$v = n \cos(\theta) - r \sin(\theta) \omega$$

4. THE BASIC JUNCTION STRUCTURE IS



5. ADD ENERGY STORAGE AND SOURCE ELEMENTS.



MULTI-BOND NOTATION

The basic bond graph notation is cumbersome for mechanisms of even modest complexity.

The more compact *multibond* notation offsets this problem.

In multibond notation, the spring pendulum model is

$$C \begin{array}{c} \diagdown \\ \hline \diagup \end{array} \begin{array}{c} \hline \\ \diagdown \\ \hline \end{array} MTF \begin{array}{c} \hline \\ \diagdown \\ \hline \end{array} \begin{array}{c} \hline \\ \diagdown \\ \hline \end{array} I$$

The multibond



depicts a multiple (or vector) of power flows
each with an associated multiple (vector) of efforts, flows, momenta,
displacements

e.g.,

$$\begin{bmatrix} x \\ y \end{bmatrix} \quad \begin{bmatrix} u \\ v \end{bmatrix} \quad \begin{bmatrix} F_x \\ F_y \end{bmatrix} \quad \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$

All associated elements are multiports

I: multiport inertia

e.g.,

$$I = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

C: multiport capacitor

e.g.,

$$C = \begin{bmatrix} 0 & 0 \\ 0 & 1/k \end{bmatrix}$$

MTF: multiport modulated transformer

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \sin(\theta) & r \cos(\theta) \\ \cos(\theta) & -r \sin(\theta) \end{bmatrix} \begin{bmatrix} n \\ \omega \end{bmatrix}$$

1 denotes a power-continuous junction where flows associated with each distinct power are identical

e.g.,

$$U_{\text{transformer}} = U_{\text{inertia}}$$

$$V_{\text{transformer}} = V_{\text{inertia}}$$