Contact instability

- Problem:
  - Contact and interaction with objects couples their dynamics into the manipulator control system
  - This change may cause instability
    - Example:
      - integral-action motion controller
      - coupling to more mass evokes instability

  - Impedance control affords a solution:
    - Make the manipulator impedance behave like a passive physical system

Hogan, N. (1988) *On the Stability of Manipulators Performing Contact Tasks,* IEEE Journal of Robotics and Automation, 4: 677-686.

Example: Integral-action motion controller

- System:
  - Mass restrained by linear spring & damper, driven by control actuator & external force

$$(ms^2 + bs + k)\, x = cu - f$$

$$\frac{x}{u} = \frac{c}{ms^2 + bs + k}$$

- Controller:
  - Integral of trajectory error

$$u = \frac{g}{s}(r - x)$$

- System + controller:

$$(ms^3 + bs^2 + ks + cg)\, x = cgr - s\, f$$

$$\frac{x}{r} = \frac{cg}{ms^3 + bs^2 + ks + cg}$$

- Isolated stability:
  - Stability requires upper bound on controller gain

$$\frac{bk}{cm} > g$$

s: Laplace variable
x: displacement variable
f: external force variable
u: control input variable
r: reference input variable
m: mass constant
b: damping constant
k: stiffness constant
c: actuator force constant
g: controller gain constant

Example (continued)

m_e: object mass constant

- Object mass: $f = m_e s^2 x$
- Coupled system: $[(m + m_e)s^3 + bs^2 + ks + cg]\, x = cgr$

$$\frac{x}{r} = \frac{cg}{(m + m_e)s^3 + bs^2 + ks + cg}$$

- Coupled stability: $bk > cg(m + m_e)$
- Choose *any* positive controller gain that will ensure isolated stability: $\frac{bk}{cm} > g$
- That controlled system is *destabilized* by coupling to a sufficiently large mass: $m_e > \frac{bk}{cg} - m$

Problem & approach

- Problem:
  - Find conditions to avoid instability due to contact & interaction
- Approach:
  - Design the manipulator controller to impose a desired interaction-port behavior
  - Describe the manipulator and its controller as an equivalent physical system
  - Find an (equivalent) physical behavior that will avoid contact/coupled instability
    - Use our knowledge of physical system behavior and how it is constrained

General object dynamics

- Assume:
  - Lagrangian dynamics
  - Passive
  - Stable in isolation

$$L(\mathbf{q}_e, \dot{\mathbf{q}}_e) = E_k^*(\mathbf{q}_e, \dot{\mathbf{q}}_e) - E_p(\mathbf{q}_e)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}_e}\right) - \frac{\partial L}{\partial \mathbf{q}_e} = \mathbf{P}_e - \mathbf{D}_e(\mathbf{q}_e, \dot{\mathbf{q}}_e)$$

- Legendre transform:
  - Kinetic co-energy to kinetic energy
  - Lagrangian form to Hamiltonian form

$$\mathbf{p}_e = \partial L / \partial \dot{\mathbf{q}}_e = \partial E_k^* / \partial \dot{\mathbf{q}}_e$$

$$E_k(\mathbf{p}_e, \mathbf{q}_e) = \mathbf{p}_e^t \dot{\mathbf{q}}_e - E_k^*(\mathbf{q}_e, \dot{\mathbf{q}}_e)$$

$$H_e(\mathbf{p}_e, \mathbf{q}_e) = \mathbf{p}_e^t \dot{\mathbf{q}}_e - L(\mathbf{q}_e, \dot{\mathbf{q}}_e)$$

$$\dot{\mathbf{q}}_e = \partial H_e / \partial \mathbf{p}_e$$

$$\dot{\mathbf{p}}_e = -\partial H_e / \partial \mathbf{q}_e - \mathbf{D}_e + \mathbf{P}_e$$

- Hamiltonian = total system energy

$$H_e(\mathbf{p}_e, \mathbf{q}_e) = E_k(\mathbf{p}_e, \mathbf{q}_e) + E_p(\mathbf{q}_e)$$

$\mathbf{q}_e$: (generalized) coordinates
L: Lagrangian
E_k^*: kinetic co-energy
E_p: potential energy
$\mathbf{D}_e$: dissipative (generalized) forces
$\mathbf{P}_e$: exogenous (generalized) forces
H_e: Hamiltonian

Sir William Rowan Hamilton

- William Rowan Hamilton
  - Born 1805, Dublin, Ireland
  - Knighted 1835
  - First Foreign Associate elected to U.S. National Academy of Sciences
  - Died 1865
- Accomplishments
  - Optics
  - Dynamics
  - Quaternions
  - Linear operators
  - Graph theory
  - …and more
    - http://www.maths.tcd.ie/pub/HistMath/People/Hamilton/

Passivity

- Basic idea: system cannot supply power indefinitely
  - Many alternative definitions, the best are energy-based
    - Wyatt et al. (1981)
- Passive: total system energy is lower-bounded
  - More precisely, *available* energy is lower-bounded
- Power flux may be positive or negative
    - Convention: power positive in
  - Power in (positive)—no limit
  - Power out (negative)—only until stored energy exhausted
    - You can store as much energy as you want but you can withdraw only what was initially stored (a finite amount)
- Passivity ≠ stability
  - Example:
    - Interaction between oppositely charged beads, one fixed, on free to move on a wire

Wyatt, J. L., Chua, L. O., Gannett, J. W., Göknar, I. C. and Green, D. N. (1981) Energy Concepts in the State-Space Theory of Nonlinear n-Ports: Part I — Passivity. IEEE Transactions on Circuits and Systems, Vol. CAS-28, No. 1, pp. 48-61.

Stability

- Stability:
  - Convergence to equilibrium
- Use Lyapunov's second method
  - A generalization of energy-based analysis
  - Lyapunov function: positive-definite non-decreasing state function
  - Sufficient condition for asymptotic stability: Negative semi-definitive rate of change of Lyapunov function
- For physical systems total energy may be a useful candidate Lyapunov function
  - Equilibria are at an energy minima
  - Dissipation $\Rightarrow$ energy reduction $\Rightarrow$ convergence to equilibrium
  - Hamiltonian form describes dynamics in terms of total energy

Steady state & equilibrium

- Steady state:
  - Kinetic energy is a positive-definite non-decreasing function of generalized momentum
- Assume:
  - Dissipative (internal) forces vanish in steady-state
    - Rules out static (Coulomb) friction
  - Potential energy is a positive-definite non-decreasing function of generalized displacement
    - Steady-state is a unique equilibrium configuration
- Steady state is equilibrium at the origin of the state space $\{\mathbf{p}_e, \mathbf{q}_e\}$

$$\dot{\mathbf{q}}_e = \mathbf{0} = \partial H_e / \partial \mathbf{p}_e = \partial E_k / \partial \mathbf{p}_e$$

$$\partial E_k / \partial \mathbf{p}_e = \mathbf{0} \Rightarrow \mathbf{p}_e = \mathbf{0}$$

$$\dot{\mathbf{p}}_e = \mathbf{0} = -\partial H_e / \partial \mathbf{q}_e - \mathbf{D}_e + \mathbf{P}_e$$

$$\text{Assume } \mathbf{D}_e(\mathbf{0}, \mathbf{q}_e) = \mathbf{0}$$

$$\text{Isolated} \Rightarrow \mathbf{P}_e = \mathbf{0}$$

$$\left.\frac{\partial H_e}{\partial \mathbf{q}_e}\right|_{\mathbf{p}_e=\mathbf{0}} = \left.\frac{\partial E_k}{\partial \mathbf{q}_e}\right|_{\mathbf{p}_e=\mathbf{0}} + \frac{\partial E_p}{\partial \mathbf{q}_e}$$

$$\left.\frac{\partial E_k}{\partial \mathbf{q}_e}\right|_{\mathbf{p}_e=\mathbf{0}} = \mathbf{0} \qquad \therefore \left.\frac{\partial H_e}{\partial \mathbf{q}_e}\right|_{\mathbf{p}_e=\mathbf{0}} = \frac{\partial E_p}{\partial \mathbf{q}_e}$$

$$\partial E_p / \partial \mathbf{q}_e = \mathbf{0} \Rightarrow \mathbf{q}_e = \mathbf{0}$$

Notation

- Represent partial derivatives using subscripts
- H_e is a scalar
  - the Hamiltonian state function
- $\mathbf{H}_{eq}$ is a vector
  - Partial derivatives of the Hamiltonian w.r.t. each element of $\mathbf{q}_e$
- $\mathbf{H}_{ep}$ is a vector
  - Partial derivatives of the Hamiltonian w.r.t. each element of $\mathbf{p}_e$

$$\mathbf{H}_{eq} = \frac{\partial H_e}{\partial \mathbf{q}_e}$$

$$\mathbf{H}_{ep} = \frac{\partial H_e}{\partial \mathbf{p}_e}$$

$$\dot{\mathbf{q}}_e = \mathbf{H}_{ep}(\mathbf{p}_e, \mathbf{q}_e)$$

$$\dot{\mathbf{p}}_e = -\mathbf{H}_{eq}(\mathbf{p}_e, \mathbf{q}_e) - \mathbf{D}_e(\mathbf{p}_e, \mathbf{q}_e) + \mathbf{P}_e$$

Isolated stability

- Use the Hamiltonian as a Lyapunov function
  - Positive-definite non-decreasing function of state
  - Rate of change of stored energy = power in – power dissipated
- Sufficient condition for asymptotic stability:
  - Dissipative generalized forces are a positive-definite function of generalized momentum
    - Dissipation may vanish if $\mathbf{p}_e = \mathbf{0}$ and system is not at equilibrium
    - But $\mathbf{p}_e = \mathbf{0}$ does not describe any system trajectory
      - LaSalle-Lefshetz theorem
  - Energy decreases on all non-equilibrium system trajectories

$$dH_e/dt = \mathbf{H}_{eq}^t \dot{\mathbf{q}}_e + \mathbf{H}_{ep}^t \dot{\mathbf{p}}_e$$

$$dH_e/dt = \mathbf{H}_{eq}^t \mathbf{H}_{ep} + \mathbf{H}_{ep}^t \left(-\mathbf{H}_{eq} - \mathbf{D}_e + \mathbf{P}_e\right)$$

$$dH_e/dt = \dot{\mathbf{q}}_e^t \mathbf{P}_e - \dot{\mathbf{q}}_e^t \mathbf{D}_e$$

$$\text{Isolated} \Rightarrow \mathbf{P}_e = \mathbf{0}$$

$$\therefore dH_e/dt = -\dot{\mathbf{q}}_e^t \mathbf{D}_e$$

$$\dot{\mathbf{q}}_e^t \mathbf{D}_e > 0 \Rightarrow dH_e/dt < 0 \quad \forall\, \mathbf{p}_e \neq \mathbf{0}$$

Physical system interaction

- Interaction of general dynamic systems
  - Many possibilities: cascade, parallel, feedback…
    - Two linear systems:

$$y_1 = G_1(s)u_1$$
$$y_2 = G_2(s)u_2$$

    - Cascade coupling equations:

$$y_3 = y_2$$
$$u_2 = y_1$$
$$u_1 = u_3$$

    - Combination:

$$y_3 = G_3(s)u_3$$
$$G_3(s) = G_2(s)G_1(s)$$

- *Not* power-continuous

$$y_3 u_3 \neq y_2 u_2 + y_1 u_1$$

- Interaction of physical systems
  - If u_i and y_i are power conjugates
  - G_i are impedances or admittances
  - Power-continuous connection:
    - Power into coupled system must equal net power into component systems

$$u_3 y_3 = u_1 y_1 + u_2 y_2$$

Interaction port

- Assume coupling occurs at a set of points on the object $\mathbf{X}_e$
  - This defines an interaction port
  - $\mathbf{X}_e$ is as a function of generalized coordinates $\mathbf{q}_e$ $$\mathbf{X}_e = \mathbf{L}_e(\mathbf{q}_e)$$
  - Generalized velocity determines port velocity $$\mathbf{V}_e = \mathbf{J}_e(\mathbf{q}_e)\dot{\mathbf{q}}_e$$
  - Port force determines generalized force $$\mathbf{P}_e = \mathbf{J}_e^t(\mathbf{q}_e)\mathbf{F}_e$$
- These relations are always well-defined
  - Guaranteed by the definition of generalized coordinates

Simple impedance

- Target (ideal) behavior of manipulator
  - Elastic and viscous behavior
- In Hamiltonian form:
  - Hamiltonian = potential energy
  - Assume $\mathbf{V}_o = \mathbf{0}$ for stability analysis
  - Isolated: $\mathbf{V}_z = \mathbf{0}$ or $\mathbf{F}_z = \mathbf{0}$
  - Sufficient condition for isolated asymptotic stability:

$$\mathbf{B}^t \dot{\mathbf{q}}_z > 0 \quad \forall \mathbf{V}_z \neq \mathbf{0}$$

- Unconstrained mass in Hamiltonian form
  - Hamiltonian = kinetic energy
  - Arbitrarily small mass
- Couple these with common velocity

$$\mathbf{F}_z = \mathbf{K}(\mathbf{X}_z - \mathbf{X}_o) + \mathbf{B}(\mathbf{V}_z)$$

$$\dot{\mathbf{p}}_z = \mathbf{H}_{zq}(\mathbf{q}_z) + \mathbf{B}(\mathbf{V}_z) \qquad \mathbf{q}_z = \mathbf{X}_z - \mathbf{X}_o$$

$$\dot{\mathbf{q}}_z = \mathbf{V}_z - \mathbf{V}_o \qquad \mathrm{H}_z(\mathbf{q}_z) = \int \mathbf{K}(\mathbf{q}_z) d\mathbf{q}_z$$

$$\mathbf{F}_z = \dot{\mathbf{p}}_z$$

$$\mathbf{V}_o = \mathbf{V}_z = \mathbf{0} \Rightarrow \mathbf{q}_z = \text{constant} \Rightarrow \mathbf{F}_z = \text{constant}$$

$$\mathbf{F}_z = \mathbf{0} \Rightarrow \mathbf{H}_{zq} = -\mathbf{B} \therefore d\mathrm{H}_z / dt = \mathbf{H}_{zq}^t \dot{\mathbf{q}}_z = -\mathbf{B}^t \dot{\mathbf{q}}_z$$

$$\dot{\mathbf{q}}_e = \mathbf{H}_{ep}(\mathbf{p}_e) \qquad \mathrm{H}_e(\mathbf{p}_e) = \tfrac{1}{2} \mathbf{p}_e^t \mathbf{M}^{-1} \mathbf{p}_e$$

$$\dot{\mathbf{p}}_e = \mathbf{F}_e$$

$$\mathbf{V}_e = \dot{\mathbf{q}}_e$$

$$\mathbf{V}_e = \mathbf{V}_z$$

$$\mathbf{F}_e^t \mathbf{V}_e + \mathbf{F}_z^t \mathbf{V}_z = \mathbf{0}$$

Mass coupled to simple impedance

- Hamiltonian form
  - Total energy = sum of components
- Assume positive-definite, non-decreasing potential energy
  - Equilibrium at $(\mathbf{p}_e, \mathbf{q}_z) = (\mathbf{0}, \mathbf{0})$

$$\mathrm{H}_t(\mathbf{p}_e, \mathbf{q}_z) = \mathrm{H}_e(\mathbf{p}_e) + \mathrm{H}_z(\mathbf{q}_z)$$

$$\dot{\mathbf{p}}_e = -\mathbf{H}_{tq}(\mathbf{q}_z) - \mathbf{B}(\mathbf{H}_{tp}(\mathbf{p}_e))$$

$$\dot{\mathbf{q}}_z = \mathbf{H}_{tp}(\mathbf{p}_e)$$

- Rate of change of Hamiltonian:

$$d\mathrm{H}_t/dt = \mathbf{H}_{tp}^t \dot{\mathbf{p}}_e + \mathbf{H}_{tq}^t \dot{\mathbf{q}}_z$$

$$d\mathrm{H}_t/dt = -\mathbf{H}_{tp}^t \mathbf{H}_{tq} - \mathbf{H}_{tp}^t \mathbf{B} + \mathbf{H}_{tq}^t \mathbf{H}_{tp} = -\dot{\mathbf{q}}_z^t \mathbf{B}$$

- Sufficient condition for asymptotic stability
  - And because mass is unconstrained, stability is global

$$\dot{\mathbf{q}}_z^t \mathbf{B} > 0 \quad \forall \mathbf{p}_e \neq \mathbf{0}$$

General object coupled to simple impedance

- Total Hamiltonian (energy) is sum of components
- Assume
  - Both systems at equilibrium
  - Interaction port positions coincide at coupling
- Total energy is a positive-definite, non-decreasing state function
- Rate of change of energy:

$$H_t(\mathbf{p}_e, \mathbf{q}_e) = H_e(\mathbf{p}_e, \mathbf{q}_e) + H_z(\mathbf{q}_z)$$

$$H_t(\mathbf{p}_e, \mathbf{q}_e) = E_k(\mathbf{p}_e, \mathbf{q}_e) + E_p(\mathbf{q}_e) + H_z(\mathbf{L}_e(\mathbf{q}_e) - \mathbf{X}_o)$$

$$dH_t/dt = \mathbf{H}_{zq}^t \mathbf{J}_e \mathbf{H}_{ep} + \mathbf{H}_{eq}^t \mathbf{H}_{ep} - \mathbf{H}_{ep}^t \mathbf{H}_{eq}$$

$$- \mathbf{H}_{ep}^t \mathbf{D}_e - \mathbf{H}_{ep}^t \mathbf{J}_e^t \mathbf{H}_{zq} - \mathbf{H}_{ep}^t \mathbf{J}_e^t \mathbf{B}$$

$$dH_t/dt = -\dot{\mathbf{q}}_e^t \mathbf{D}_e - \dot{\mathbf{q}}_z^t \mathbf{B}$$

- The previous conditions sufficient for stability of
  - Object in isolation
  - Simple impedance coupled to arbitrarily small mass
- …ensure global asymptotic coupled stability
  - Energy decreases on all non-equilibrium state trajectories
  - True for objects of arbitrary dynamic order

Simple impedance controller implementation

- Robot model:
  - Inertial mechanism, statically balanced (or zero gravity), effort-controlled actuators
    - Hamiltonian = kinetic energy
- Controller:
  - Transform simple impedance to manipulator configuration space
- Controller coupled to robot:
  - Same structure as a physical system with Hamiltonian H_c

$$H_c = H_m + H_z$$

$$\dot{\mathbf{q}}_m = \mathbf{H}_{mp} \qquad H_m = \tfrac{1}{2}\mathbf{p}_m^t \mathbf{I}^{-1}(\mathbf{q}_m)\mathbf{p}_m$$

$$\dot{\mathbf{p}}_m = -\mathbf{H}_{mq} - \mathbf{D}_m + \mathbf{P}_a + \mathbf{J}_m^t \mathbf{F}_m$$

$$\mathbf{V}_m = \mathbf{J}_m \dot{\mathbf{q}}_m$$

$$\mathbf{X}_m = \mathbf{L}_m(\mathbf{q}_m)$$

$$\mathbf{P}_a = -\mathbf{J}_m^t \{\mathbf{K}(\mathbf{L}_m(\mathbf{q}_m) - \mathbf{X}_o) - \mathbf{B}(\mathbf{J}_m \dot{\mathbf{q}}_m)\}$$

$$\dot{\mathbf{q}}_m = \mathbf{H}_{cp}$$

$$\dot{\mathbf{p}}_m = -\mathbf{H}_{cq} - \mathbf{D}_m - \mathbf{J}_m^t \mathbf{B} + \mathbf{J}_m^t \mathbf{F}_m$$

$$\mathbf{V}_m = \mathbf{J}_m \dot{\mathbf{q}}_m$$

$$\mathbf{X}_m = \mathbf{L}_m(\mathbf{q}_m)$$

$\mathbf{q}_m$: generalized coordinates (configuration variables)
$\mathbf{p}_m$: generalized momenta
H_m: Hamiltonian
$\mathbf{I}$: inertia
$\mathbf{D}_m$: dissipative (generalized) forces
$\mathbf{P}_a$: actuator (generalized) forces
$\mathbf{X}_m, \mathbf{V}_m, \mathbf{F}_m$: interaction port position, velocity, force
$\mathbf{L}_m, \mathbf{J}_m$: kinematic equations, Jacobian

Simple impedance controller isolated stability

- Rate of change of Hamiltonian:

$$dH_c/dt = \mathbf{H}_{cq}^t\mathbf{H}_{cp} - \mathbf{H}_{cp}^t\mathbf{H}_{cq} - \mathbf{H}_{cp}^t\mathbf{D}_m - \mathbf{H}_{cp}^t\mathbf{J}_m^t\mathbf{B} + \mathbf{H}_{cp}^t\mathbf{J}_m^t F_m$$

$$dH_c/dt = -\dot{\mathbf{q}}_m^t\mathbf{D}_m - \mathbf{V}_m^t\mathbf{B} + \mathbf{V}_m^t\mathbf{F}_m$$

$$\mathbf{F}_m = \mathbf{0} \Rightarrow dH_c/dt = -\dot{\mathbf{q}}_m^t\mathbf{D}_m - \mathbf{V}_m^t\mathbf{B}$$

- Energy decreases on all non-equilibrium trajectories if
  - System is isolated $\mathbf{F}_m = \mathbf{0}$
  - Dissipative forces are positive-definite $\dot{\mathbf{q}}_m^t\mathbf{D}_m > 0, \mathbf{V}_m^t\mathbf{B} > 0 \quad \forall \mathbf{p}_m \neq \mathbf{0}$
- Minimum energy is at $\mathbf{q}_z = \mathbf{0}, \mathbf{X}_m = \mathbf{X}_o$
  - But this may not define a unique manipulator configuration
  - Hamiltonian is a positive-definite non-decreasing function of $\mathbf{q}_z$ but usually *not* of configuration $\mathbf{q}_m$
- Interaction-port impedance may not control internal degrees of freedom
  - Could add terms to controller but for simplicity…
- Assume:
  - Non-redundant mechanism
  - Non-singular Jacobian
- Then
  - Hamiltonian is positive-definite & non-decreasing in a region about $\mathbf{q}_m = \mathbf{L}^{-1}(\mathbf{X}_o)$
- *Local* asymptotic stability

Simple impedance controller coupled stability

- Coupling kinematics $\mathbf{q}_t = \mathbf{q}_t(\mathbf{q}_m, \mathbf{q}_e)$
  - Coupling relates $\mathbf{q}_m$ to $\mathbf{q}_e$ but no need to solve explicitly
  - Total Hamiltonian (energy) is sum of components $H_t = H_e(\mathbf{p}_e, \mathbf{q}_e) + H_c(\mathbf{p}_m, \mathbf{q}_m)$
- Rate of change of Hamiltonian

$$dH_t/dt = \mathbf{H}_{eq}^t \mathbf{H}_{ep} + \mathbf{H}_{ep}^t\left(-\mathbf{H}_{eq} - \mathbf{D}_e + \mathbf{J}_e^t \mathbf{F}_e\right) + \mathbf{H}_{cq}^t \mathbf{H}_{cp} + \mathbf{H}_{cp}^t\left(-\mathbf{H}_{cq} - \mathbf{D}_m - \mathbf{J}_m^t \mathbf{B} + \mathbf{J}_m^t \mathbf{F}_m\right)$$

$$dH_t/dt = -\dot{\mathbf{q}}_e^t \mathbf{D}_e + \dot{\mathbf{q}}_e^t \mathbf{J}_e^t \mathbf{F}_e - \dot{\mathbf{q}}_m^t\left(\mathbf{D}_m + \mathbf{J}_m^t \mathbf{B}\right) + \dot{\mathbf{q}}_m^t \mathbf{J}_m^t \mathbf{F}_m$$

$$dH_t/dt = -\dot{\mathbf{q}}_e^t \mathbf{D}_e + \mathbf{V}_e^t \mathbf{F}_e - \dot{\mathbf{q}}_m^t \mathbf{D}_m - \mathbf{V}_m^t \mathbf{B} + \mathbf{V}_m^t \mathbf{F}_m$$

- Coupling cannot generate power

$$\mathbf{V}_e^t \mathbf{F}_e + \mathbf{V}_m^t \mathbf{F}_m = 0$$

$$\therefore dH_t/dt = -\dot{\mathbf{q}}_e^t \mathbf{D}_e - \dot{\mathbf{q}}_m^t \mathbf{D}_m - \mathbf{V}_m^t \mathbf{B}$$

- The previous conditions sufficient for stability of
  - Object in isolation
  - Simple impedance controlled robot
- …ensure local asymptotic coupled stability

Kinematic errors

- Assume controller and interaction port kinematics differ
  - Controller kinematics maps configuration to a point $\tilde{\mathbf{X}}$
  - Corresponding potential function is positive-definite, non-decreasing in a region about $\tilde{\mathbf{q}}_m = \tilde{\mathbf{L}}^{-1}(\mathbf{X}_o)$
- Assume self-consistent controller kinematics
  - the (erroneous) Jacobian is the correct derivative of the (erroneous) kinematics

$$\mathbf{P}_a = -\tilde{\mathbf{J}}^t\left\{\mathbf{K}\left(\tilde{\mathbf{L}}(\mathbf{q}_m) - \mathbf{X}_o\right) - \mathbf{B}\left(\tilde{\mathbf{J}}\dot{\mathbf{q}}_m\right)\right\}$$

$$\tilde{\mathbf{X}} = \tilde{\mathbf{L}}(\mathbf{q}_m) \neq \mathbf{L}_m(\mathbf{q}_m)$$

$$\tilde{\mathrm{H}}_z(\mathbf{q}_m) = \mathrm{H}_z(\tilde{\mathbf{q}}_z) = \mathrm{H}_z\left(\tilde{\mathbf{L}}(\mathbf{q}_m) - \mathbf{X}_o\right)$$

$$\partial\tilde{\mathbf{L}}/\partial\mathbf{q}_m = \tilde{\mathbf{J}}$$

$$d\tilde{\mathbf{X}}/dt = \tilde{\mathbf{V}} = \tilde{\mathbf{J}}(\mathbf{q}_m)\dot{\mathbf{q}}_m$$

$$d\tilde{\mathrm{H}}_z/dt = \mathbf{H}_{zq}^t \frac{\partial\tilde{\mathbf{L}}}{\partial\mathbf{q}_m}\dot{\mathbf{q}}_m = \mathbf{H}_{zq}^t\tilde{\mathbf{J}}\dot{\mathbf{q}}_m = \mathbf{H}_{zq}^t\tilde{\mathbf{V}}$$

Kinematic errors (continued)

- Hamiltonian of this controller coupled to the robot

$$\tilde{H}_c(\mathbf{p}_m, \mathbf{q}_m) = H_m(\mathbf{p}_m, \mathbf{q}_m) + H_z(\tilde{\mathbf{q}}_z)$$

$$\tilde{H}_c(\mathbf{p}_m, \mathbf{q}_m) = H_m(\mathbf{p}_m, \mathbf{q}_m) + H_z\left(\tilde{\mathbf{L}}(\mathbf{q}_m) - \mathbf{X}_o\right)$$

  - Hamiltonian state equations

$$\dot{\mathbf{q}}_m = \mathbf{H}_{mp}$$

$$\dot{\mathbf{p}}_m = -\mathbf{H}_{mq} - \mathbf{D}_m - \tilde{\mathbf{J}}^t \mathbf{H}_{zq} - \tilde{\mathbf{J}}^t \mathbf{B} + \mathbf{J}_m^t \mathbf{F}_m$$

  - Rate of change of the Hamiltonian

$$d\tilde{H}_c / dt = \mathbf{H}_{zq}^t \tilde{\mathbf{J}} \mathbf{H}_{mp} + \mathbf{H}_{mq}^t \mathbf{H}_{mp} + \mathbf{H}_{mp}^t\left(-\mathbf{H}_{mq} - \mathbf{D}_m - \tilde{\mathbf{J}}^t \mathbf{H}_{zq} - \tilde{\mathbf{J}}^t \mathbf{B} + \mathbf{J}_m^t \mathbf{F}_m\right)$$

$$d\tilde{H}_c / dt = -\dot{\mathbf{q}}_m^t \mathbf{D}_m - \tilde{\mathbf{V}}^t \mathbf{B} + \mathbf{J}_m^t \mathbf{F}_m$$

  - In isolation

$$\mathbf{F}_m = \mathbf{0} \Rightarrow d\tilde{H}_c / dt = -\dot{\mathbf{q}}_m^t \mathbf{D}_m - \tilde{\mathbf{V}}^t \mathbf{B}$$

- Previous conditions on $\mathbf{D}_m$ & $\mathbf{B}$ are sufficient for isolated local asymptotic stability

Insensitivity to kinematic errors

- The same conditions are also sufficient to ensure local asymptotic coupled stability
  - Coupled system Hamiltonian and its rate of change:
- Stability properties are insensitive to kinematic errors
  - Provided they are self-consistent
- Note that these results do not require small kinematic errors
  - Could arise when contact occurs at unexpected locations
  - e.g., on the robot links rather than the end-point

$$\tilde{H}_t = E_k(\mathbf{p}_e, \mathbf{q}_e) + E_p(\mathbf{q}_e) +$$
$$H_m(\mathbf{p}_m, \mathbf{q}_m) + H_z(\tilde{\mathbf{L}}(\mathbf{q}_m) - \mathbf{X}_o)$$
$$d\tilde{H}_t / dt = -\dot{\mathbf{q}}_e^t \mathbf{D}_e - \dot{\mathbf{q}}_m^t \mathbf{D}_m - \tilde{\mathbf{V}}^t \mathbf{B}$$

Parallel & feedback connections

- Power continuity

$$y_3u_3 = y_2u_2 + y_1u_1$$

- Parallel connection equations
- Power balance

—OK

$$y_3 = \pm y_2 \pm y_1$$
$$u_3 = u_2 = u_1$$
$$y_3u_3 = \pm y_2u_2 \pm y_1u_1$$

- Feedback connection equations
- Power balance

—OK

$$y_3 = y_1 = u_2$$
$$u_1 = u_3 - y_2$$
$$u_1y_1 = u_3y_3 - y_2u_2$$

Summary remarks

- Interaction stability
  - The above results can be extended
    - Neutrally stable objects
    - Kinematic constraints
      - no dynamics
    - Interface dynamics
      - e.g., due to sensors
  - A “simple” impedance can provide a robust solution to the contact instability problem
- Structure matters
  - Dynamics of physical systems are constrained in useful ways
- It may be beneficial to *impose* physical system structure on a general dynamic system
  - e.g. a robot controller

Some other Irishmen of note

- Bishop George **Berkeley**
- Robert **Boyle**
- John Boyd **Dunlop**
- George Francis **Fitzgerald**
- William Rowan **Hamilton**
- William Thomson (Lord **Kelvin**)
- Joseph **Larmor**
- Charles **Parsons**
- Osborne **Reynolds**
- George Gabriel **Stokes**