2.094

FINITE ELEMENT ANALYSIS OF SOLIDS AND FLUIDS

SPRING 2008

Homework 8 - Solution

		Assigned:	04/10/2008
Instructor:	Prof. K. J. Bathe	Due:	04/17/2008

Problem 1 (20 points):

Since $H, h \ll b$, we only consider the displacement u in the x_1-direction with the plane stress assumption.

Total Lagrangian formulation

Applied force: ${}^t f^B = {}^t\rho\, {}^t x_1 \omega^2$.

Thickness: ${}^0t = H\left(\dfrac{{}^0x_1 - b}{a-b}\right) + h\left(\dfrac{{}^0x_1 - a}{b-a}\right)$, ${}^tt = H\left(\dfrac{{}^tx_1 - {}^tb}{{}^ta - {}^tb}\right) + h\left(\dfrac{{}^tx_1 - {}^ta}{{}^tb - {}^ta}\right)$

The displacement field with a single two node element is

$$u = h_1(r)u^1 + h_2(r)u^2 = \frac{1}{2}(1+r)u^1 + \frac{1}{2}(1-r)u^2 = \begin{bmatrix} \frac{1}{2}(1+r) & \frac{1}{2}(1-r) \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$$

The Jacobian matrices are ${}^0J^{-1} = \dfrac{\partial r}{\partial {}^0x_1} = \dfrac{2}{b-a}$ and ${}^tJ^{-1} = \dfrac{\partial r}{\partial {}^tx_1} = \dfrac{2}{{}^tb - {}^ta}$

The linear strain components can be written as

$$\begin{aligned}
{}_0e_{11} &= \frac{\partial u}{\partial {}^0x_1} + \frac{\partial {}^tu}{\partial {}^0x_1}\frac{\partial u}{\partial {}^0x_1} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial {}^0x_1} + \left(\frac{\partial {}^tu}{\partial r}\frac{\partial r}{\partial {}^0x_1}\right)\left(\frac{\partial u}{\partial r}\frac{\partial r}{\partial {}^0x_1}\right) \\
&= {}^0J^{-1}\frac{\partial u}{\partial r} + \left({}^0J^{-1}\right)^2 \frac{\partial {}^tu}{\partial r}\frac{\partial u}{\partial r} \\
&= \begin{bmatrix} \left\{ {}^0J^{-1} + \left({}^0J^{-1}\right)^2 \dfrac{\partial {}^tu}{\partial r} \right\} h_{1,r} & \left\{ {}^0J^{-1} + \left({}^0J^{-1}\right)^2 \dfrac{\partial {}^tu}{\partial r} \right\} h_{2,r} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}
\end{aligned}$$

$$
{}_0e_{33} = \frac{u}{{}^0x_1} + \frac{\left({}^tu\right)u}{{}^0x_1^2} = \begin{bmatrix} \frac{h_1}{{}^0x_1} + \frac{\left({}^tu\right)h_1}{{}^0x_1^2} & \frac{h_2}{{}^0x_1} + \frac{\left({}^tu\right)h_2}{{}^0x_1^2} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}
$$

where

$$
\frac{\partial\,{}^tu}{\partial r} = h_{1,r}\,{}^tu^1 + h_{2,r}\,{}^tu^2\,,\quad {}^tu = h_1\,{}^tu^1 + h_2\,{}^tu^2 \text{ and } {}^0x_1 = \frac{b-a}{2}r + \frac{b+a}{2}
$$

Therefore,

$$
{}_0^t\underline{B}_L = \begin{bmatrix} \left\{ {}^0J^{-1} + \left({}^0J^{-1}\right)^2 \frac{\partial\,{}^tu}{\partial r} \right\} h_{1,r} & \left\{ {}^0J^{-1} + \left({}^0J^{-1}\right)^2 \frac{\partial\,{}^tu}{\partial r} \right\} h_{2,r} \\ \frac{h_1}{{}^0x_1} + \frac{\left({}^tu\right)h_1}{{}^0x_1^2} & \frac{h_2}{{}^0x_1} + \frac{\left({}^tu\right)h_2}{{}^0x_1^2} \end{bmatrix}
$$

The nonlinear strain components are

$$
{}_0\eta_{11} = \frac{1}{2}\left(\frac{\partial u}{\partial\,{}^0x_1}\right)^2 \rightarrow \delta\,{}_0\eta_{11} = \frac{\partial \delta u}{\partial\,{}^0x_1}\frac{\partial u}{\partial\,{}^0x_1} = \left({}^0J^{-1}\frac{\partial \delta u}{\partial r}\right)\left({}^0J^{-1}\frac{\partial u}{\partial r}\right)
$$

$$
{}_0\eta_{33} = \frac{1}{2}\left(\frac{u}{{}^0x_1}\right)^2 \rightarrow \delta\,{}_0\eta_{33} = \frac{\delta u}{{}^0x_1}\frac{u}{{}^0x_1}
$$

Then,

$$
\begin{aligned}
{}_0^tS_{11}\delta\,{}_0\eta_{11} &= {}_0^tS_{11}\left({}^0J^{-1}\frac{\partial \delta u}{\partial r}\right)\left({}^0J^{-1}\frac{\partial u}{\partial r}\right) \\
&= \begin{bmatrix} \delta u^1 & \delta u^2 \end{bmatrix} \begin{bmatrix} {}^0J^{-1}h_{1,r} \\ {}^0J^{-1}h_{2,r} \end{bmatrix} {}_0^tS_{11} \begin{bmatrix} {}^0J^{-1}h_{1,r} & {}^0J^{-1}h_{2,r} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
{}_0^tS_{33}\delta\,{}_0\eta_{33} &= {}_0^tS_{33}\left(\frac{\delta u}{{}^0x_1}\right)\left(\frac{u}{{}^0x_1}\right) \\
&= \begin{bmatrix} \delta u^1 & \delta u^2 \end{bmatrix} \begin{bmatrix} \frac{h_1}{{}^0x_1} \\ \frac{h_2}{{}^0x_1} \end{bmatrix} {}_0^tS_{33} \begin{bmatrix} \frac{h_1}{{}^0x_1} & \frac{h_2}{{}^0x_1} \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \end{bmatrix}
\end{aligned}
$$

Therefore,

$$ {}_0^t\underline{B}_{NL} = \begin{bmatrix} {}^0J^{-1}h_{1,r} & {}^0J^{-1}h_{2,r} \\ \dfrac{h_1}{{}^0x_1} & \dfrac{h_2}{{}^0x_1} \end{bmatrix} \text{ with } {}_0^t\underline{S} = \begin{bmatrix} {}_0^tS_{11} & 0 \\ 0 & {}_0^tS_{33} \end{bmatrix} $$

Hence the final FE equation is

$$ \left[\int_{-1}^{+1} {}_0^t\underline{B}_L^T \, {}_0\underline{C} \, {}_0^t\underline{B}_L (2\pi)\left({}^0x_1\right)\left({}^0t\right)\left({}^0J^{-1}\right) dr + \int_{-1}^{+1} {}_0^t\underline{B}_{NL}^T \, {}_0^t\underline{S} \, {}_0^t\underline{B}_{NL} (2\pi)\left({}^0x_1\right)\left({}^0t\right)\left({}^0J^{-1}\right) dr\right]\begin{bmatrix} u^1 \\ u^2 \end{bmatrix} $$

$$ = \int_{-1}^{+1} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} {}^{t+\Delta t}\rho\omega^2 \left({}^{t+\Delta t}x_1\right)(2\pi)\left({}^{t+\Delta t}x_1\right)\left({}^{t+\Delta t}t\right)\left({}^{t+\Delta t}J^{-1}\right) dr - \int_{-1}^{+1} {}_0^t\underline{B}_L^T \, {}_0^t\hat{\underline{S}} (2\pi)\left({}^0x_1\right)\left({}^0t\right)\left({}^0J^{-1}\right) dr $$

Problem 2 (20 points):

Let's consider u_1^1 only (Other DOFs can be set to zero because we are interested in components corresponding to u_1^1). Then the displacement field is

$$ u_1 = \frac{1}{4}\left(1+\frac{{}^0x_1}{3}\right)\left(1+\frac{{}^0x_2}{2}\right)u_1^1 \text{ and } u_2 = 0 $$

$$ {}^tu_1 = \frac{1}{2}\left(1+\frac{{}^0x_1}{3}\right)(1.5) \text{ and } {}^tu_2 = \frac{1}{2}\left(1+\frac{{}^0x_2}{2}\right)(0.5) $$

The linear strain components are

$$ {}_0e_{11} = {}_0u_{1,1} + {}_0^tu_{k,1} \, {}_0u_{k,1} = \frac{5}{48}\left(1+\frac{{}^0x_2}{2}\right)u_1^1 $$

$$ {}_0e_{22} = {}_0u_{2,2} + {}_0^tu_{k,2} \, {}_0u_{k,2} = 0 $$

$$ {}_0e_{12} = \frac{1}{2}\left({}_0u_{1,2} + {}_0u_{2,1} + {}_0^tu_{k,1} \, {}_0u_{k,2} + {}_0^tu_{k,2} \, {}_0u_{k,1}\right) = \frac{5}{64}\left(1+\frac{{}^0x_1}{3}\right)u_1^1 $$

The nonlinear strain components are

$$\delta\,{}_0\eta_{11} = \left({}_0u_{k,1}\right)\left(\delta\,{}_0u_{k,1}\right) = \delta u_1^1 \left\{ \frac{1}{12}\left(1+\frac{{}^0x_2}{2}\right) \right\}^2 u_1^1$$

$$\delta\,{}_0\eta_{22} = \left({}_0u_{k,2}\right)\left(\delta\,{}_0u_{k,2}\right) = \delta u_1^1 \left\{ \frac{1}{8}\left(1+\frac{{}^0x_1}{3}\right) \right\}^2 u_1^1$$

$$\begin{aligned}\delta\,{}_0\eta_{12} &= \frac{1}{2}\left\{\left({}_0u_{k,1}\right)\left(\delta\,{}_0u_{k,2}\right)+\left({}_0u_{k,2}\right)\left(\delta\,{}_0u_{k,1}\right)\right\} \\ &= \frac{1}{2}\left\{\delta u_1^1 \frac{1}{8}\left(1+\frac{{}^0x_1}{3}\right)\frac{1}{12}\left(1+\frac{{}^0x_2}{2}\right)u_1^1 + \delta u_1^1 \frac{1}{12}\left(1+\frac{{}^0x_2}{2}\right)\frac{1}{8}\left(1+\frac{{}^0x_1}{3}\right)u_1^1\right\} \\ &= \delta u_1^1 \frac{1}{96}\left(1+\frac{{}^0x_1}{3}\right)\left(1+\frac{{}^0x_2}{2}\right)u_1^1\end{aligned}$$

And the stress matrices and the constitutive law are

$${}_0^t\underline{S} = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 60 & 0 \\ 0 & 0 & 0 \end{bmatrix},\quad {}_0^t\underline{\hat{S}} = \begin{bmatrix} 100 \\ 60 \\ 0 \end{bmatrix} \text{ and } {}_0\underline{C} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Therefore,

$$\left({}_0^t\underline{K}_L\right)_{11} = \int_{-3}^{3}\int_{-2}^{2}\left[\frac{5}{48}\left(1+\frac{{}^0x_2}{2}\right) \quad 0 \quad 2\cdot\frac{5}{64}\left(1+\frac{{}^0x_1}{3}\right)\right]{}_0\underline{C}\begin{bmatrix} \frac{5}{48}\left(1+\frac{{}^0x_2}{2}\right) \\ 0 \\ 2\cdot\frac{5}{64}\left(1+\frac{{}^0x_1}{3}\right) \end{bmatrix} h\,d\,{}^0x_1 d\,{}^0x_2 = 0.682Eh$$

$$\left({}_0^t\underline{K}_{NL}\right)_{11} = \int_{-3}^{3}\int_{-2}^{2}\left[\left\{\frac{1}{12}\left(1+\frac{{}^0x_2}{2}\right)\right\}^2 {}_0^tS_{11} + \left\{\frac{1}{8}\left(1+\frac{{}^0x_1}{3}\right)\right\}^2 {}_0^tS_{22}\right] h\,d\,{}^0x_1 d\,{}^0x_2 = 52.2h$$

$$\left({}_0^t\underline{F}\right)_1 = \int_{-3}^{3}\int_{-2}^{2}\left[\frac{5}{48}\left(1+\frac{{}^0x_2}{2}\right) \quad 0 \quad 2\cdot\frac{5}{64}\left(1+\frac{{}^0x_1}{3}\right)\right]\begin{bmatrix} 100 \\ 60 \\ 0 \end{bmatrix} h\,d\,{}^0x_1 d\,{}^0x_2 = 250h$$

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